General Theory of Force
And Mass Reductions with
Examples from
Existing Machines

Mechanical Engineering

B. S.

1 9 0 7

GENERAL THEORY OF FORCE AND MASS REDUCTIONS WITH EXAMPLES FROM EXISTING MACHINES

BY

Alwin L Schaller

THESIS FOR THE DEGREE OF BACHELOR OF SCIENCE

IN MECHANICAL ENGINEERING

UNIVERSITY OF ILLINOIS

June 1, 1907

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

ALWIN L SCHALLER

ENTITLED GENERAL THEORY OF FORCE AND MASS REDUCTIONS

WITH EXAMPLES FROM EXISTING MACHINES

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Science in Mechanical Engineering

L. P. Breckenridge

HEAD OF DEPARTMENT OF Mechanical Engineering

Contents.

Page.

The General Theory of Force and Mass Reduction with Examples from Existing Machines.

I - Introduction.

In many problems involving the kinetics of mechanisms it is often necessary to replace all masses by an equivalent mass concentrated at some particular point and to replace all forces by equivalent forces having chosen lines of action. It is the object of this thesis to develope equations that may be employed for making such reductions and to apply these equations to some existing machine. Considerations not directly connected with the subject but having an important bearing upon its development are also taken up; As for example, a discussion of kinetic energy and the determination of inertia forces. The theory is applied to one of the fly ball governors in

the laboratory. In order to make the subject complete the derivation of the equation of motion of the fly-ball governor is included. The application has been limited to mechanisms without springs but the equations are of a general nature and may be applied without difficulty to this class of machines.

Many notations were used throughout the discussion and it was found convenient to include them here to save repeating them whenever necessary.

Notation.

M = mass.

F = force.

a = linear acceleration.

g = acceleration due to gravity.

V = volume.

I = principal moment of inertia.

I' = moment of inertia with respect to axis of rotation.

ΔM = element of mass.

k = principal radius of gyration.

T = time of oscillation of a pendulum.

v = velocity.

$\dfrac{d\theta}{dt}$ = angular velocity.

ω = angular velocity.

$\dfrac{dx}{dt}$ = linear velocity.

α = angular acceleration.

$\mathcal{M}$ = reduced mass.

$\mathcal{F}$ = reduced force.

W = work.

II - _Mass Reduction_

1 - When masses are reduced from one point of a system to another, the dynamic conditions must remain the same, that is, the velocities and accelerations of the moving masses must remain unchanged by the reduction. In order that this condition may obtain, the reduced system at any instant must be equal to the total kinetic energy of all of the separate masses moving at that instant.

2 - _Mass_, _Moment of Inertia_, _etc_. - Before passing to the methods of reducing masses, a general discussion of mass and its properties will be given. By the term, mass, is meant the amount of matter in a body, independent of place. Mass is, therefore, distinct from weight which is merely the attraction due to gravity, and varies

from place to place. For measuring
mass, Newton's Second Law of Motion
is employed. "The acceleration of a body
is directly proportional to the force ap-
plied and inversely to the mass." This
law stated in the form of an equation
is

$$F \propto Ma \quad or \quad F = kMa$$

where k = constant. The value of the
constant k depends upon the units em-
ployed for expressing the remaining terms
of the equation. The simplest system
of units would be one in which k would
reduce to 1 (or unity) and the equation
would then read

$$F = Ma.$$

In the Engineers System (f.,p.,s.)
the units are foot, pound and second.
The pound used here is a unit of
force. From these units, the unit of
acceleration at once is ft./sec². It
now remains to deduce such a unit
of mass as to make k equal to 1.
Consider a body under the action of
gravity of weight W. The force act-

ing in this case is simply the weight.
If the acceleration is denoted by g,
we have therefore

$$W = Mg \quad \text{or} \quad M = \frac{W}{g} \quad \text{(If M is in proper units)}$$

We now have the important relation that
the mass is equal to the weight divid-
ed by the acceleration due to gravity.
The unit of mass then is

$$\frac{F}{ft/sec^2} = \frac{F \, sec^2}{ft.}$$

Having found the units we can check
our equations by substituting in them,
these fundamental units and noting
whether both members are of the same
kind.

$$F = Ma$$
$$p = \frac{p \, sec^2}{ft.} \times \frac{ft.}{sec^2}$$
$$p = p.$$

We have now to find a numerical
value for g. This has been done
experimentally both with falling bo-
dies and with the pendulum. An
average value is $32.2 \frac{ft.}{sec^2}$. The Engi-
neers System of units will be em-

ployed throughout this discussion.
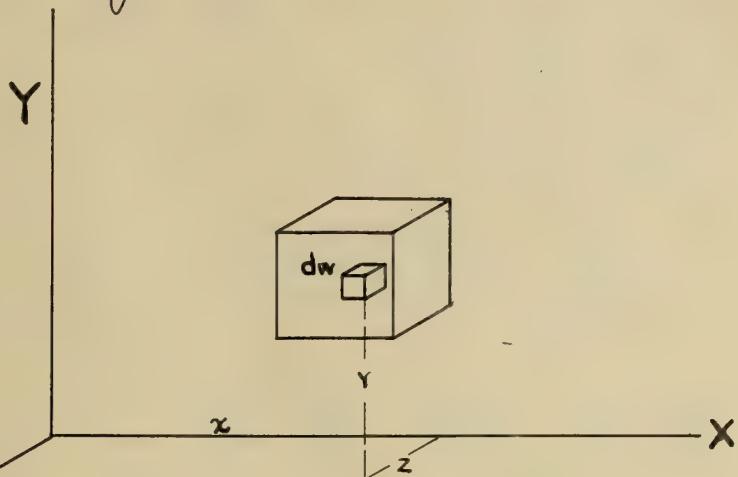
Fig 1

Consider any body of weight W as shown in figure 1. Take any particle, d W, whose coordinates are x, y and z. Gravity acts upon this element with a force equal to the weight of the element. The weights of all the elements of the body form a system of parallel forces. The resultant of these forces passes through a fixed point with reference to the body independent of its position. This point is called the center of gravity. To locate this point, use is made of the following principle. The weight of a body multiplied by the distance of its

center of gravity from any datum plane is equal to the sum of all the elements of that body multiplied by their coordinates with respect to the same plane. The moment of any particle with respect to the YZ plane is

$$dM_{yz} = x\,dw$$

and if $\bar{x}$ is coordinate of the center of gravity from the same plane, we have

$$W\bar{x} = \Sigma\,x\,dw$$
$$\bar{x} = \frac{\Sigma\,x\,dw}{W} \qquad (1)$$

The limits of the summation are such as to include the bounding surfaces of the body. In the same way, we have

$$\bar{y} = \frac{\Sigma\,y\,dw}{W}$$
$$\bar{z} = \frac{\Sigma\,z\,dw}{W}$$

The center of gravity may therefore be determined by integration when the equation of the bounding surface is given.

Let K = weight per unit volume

Then $VK = W$ and $K\,dv = dw$

Substituting in (1)

$$\bar{x} = \frac{K\int x\,dv}{KV} = \frac{\int x\,dv}{V}$$

This equation may now be integrated when the limits are known.

Most bodies with which we have to deal, however, are of such irregular outline that no equations can be given for the bounding surfaces. Any machine link will serve for illustration. In such cases, the center of gravity must be determined experimentally. In fact, the given equation is of greater value in defining the center of gravity and gaining a clear conception of what it is than in actually locating it in machine links. One method of finding the center of gravity is to suspend the body by a cord. The direction of the cord is then marked on the body, and the operation repeated with a different point of suspension. The center of gravity will lie in each one of these lines and evidently at the point of their intersection.

Figure 2   illustrates this procedure.
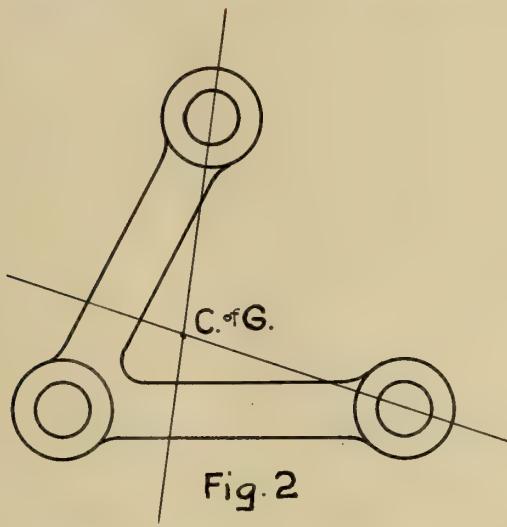
Fig. 2

The second method is to balance the
body on a knife edge, the vertical plane
of which is indicated on the body. This
operation is repeated for two other posi-
tions; and, as the center of gravity lies
in the vertical plane of the straight edge,
it is in the point of intersection of the
three planes. Another method is to
rest one end of the body on a fixed
knife edge and suspend the other
with a spring balance, as shown
in figure 3.
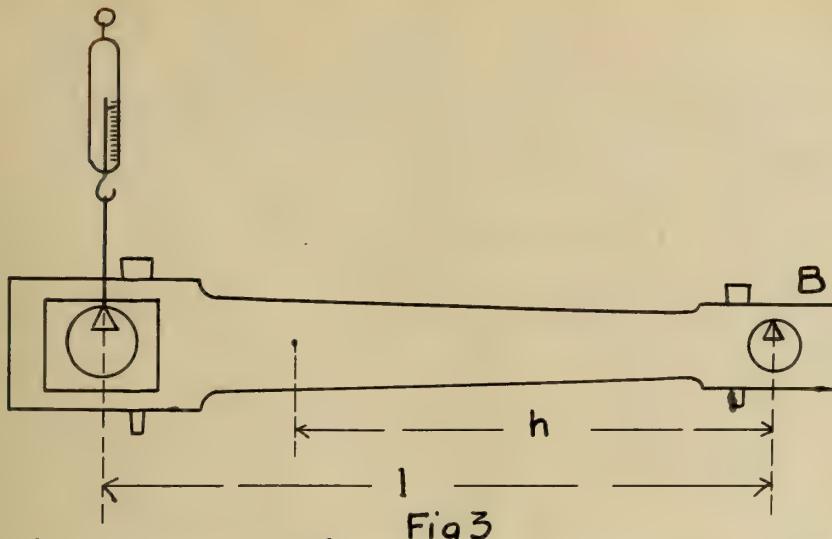
Fig 3

Let F = force on scale

W = weight of link

l = distance between knife edges.

Taking moments about B, we have.

$$Fl = Wh \quad \text{or} \quad h = \frac{Fl}{W}$$

In a preceding paragraph it was shown that the weight and mass of a body had a fixed relation, viz:

$$M = \frac{W}{g}$$

Substituting Mg for W in the following equation

$$\bar{x} = \frac{\Sigma x \, dW}{W} = \frac{g\Sigma x \, dM}{Mg} = \frac{\Sigma x \, dM}{M}$$

This proves the important relation that the center of mass coincides with the center of gravity.
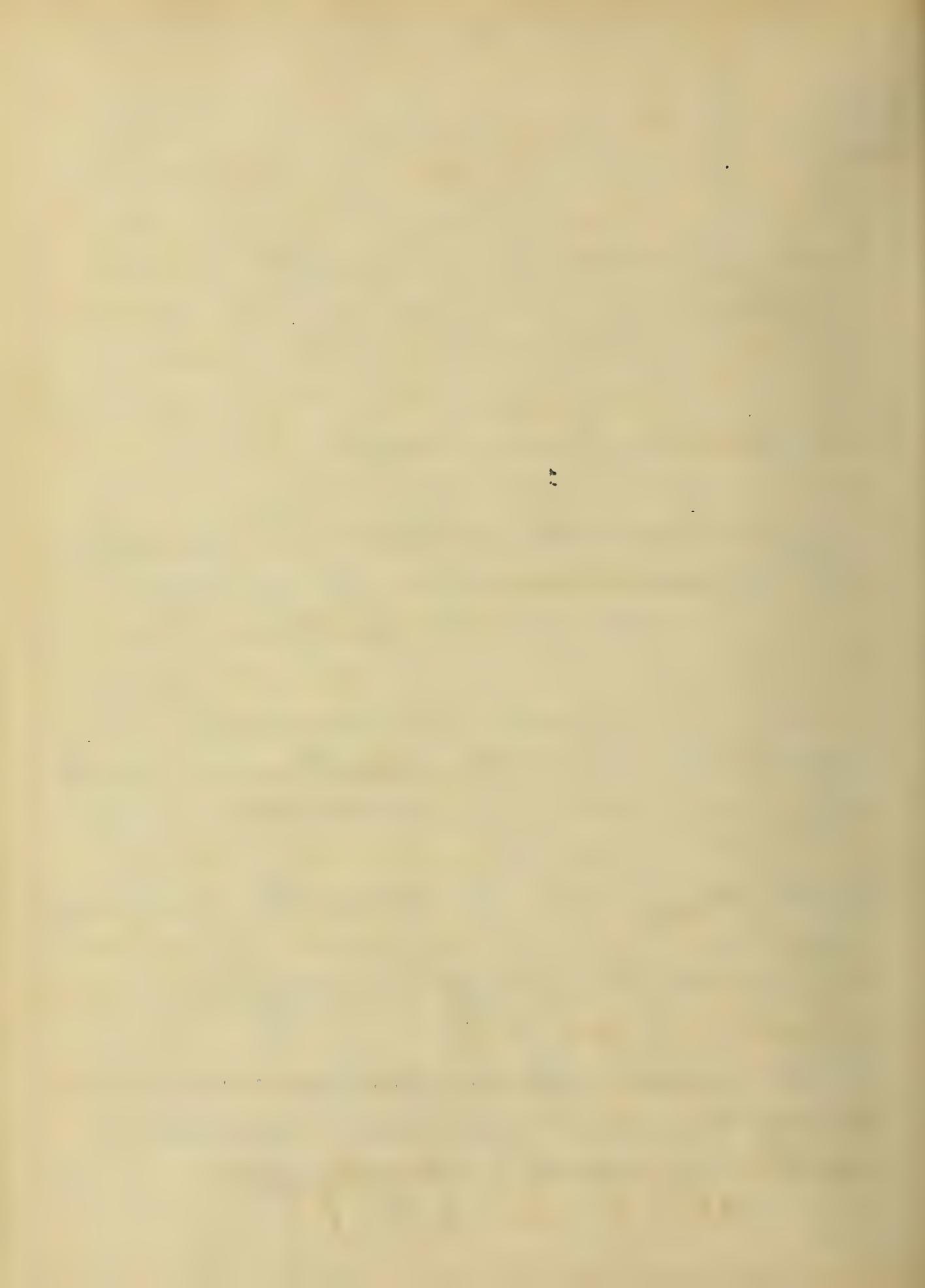
In the study of rotating masses, the expression $\Sigma r^2 dM$ is often met with. To this expression Euler gave the name, moment of inertia. The definition of moment of inertia of a body is therefore the sum of the products obtained by multiplying each element of mass by the square of its distance from a fixed axis.

To find the moment of inertia, proceed as follows:

Let ρ = distance of any element.

γ = density

By definition

$$I = \Sigma \, dM \rho^2$$

But $dM = \gamma \, dV$

$$\therefore I = \gamma \Sigma \, dV \rho^2$$

The limits of integration must be such so as to include all elements of the body.

Fig. 4.

We may express the moment of inertia as the product of the whole mass by the square of a length as

$$M k^2 = I \quad \text{or} \quad k = \sqrt{\frac{I}{M}}$$
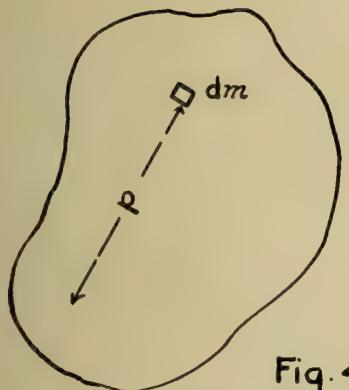
This length is known as the radius of gyration.

An example will now be given illustrating the method of finding the moment of inertia of a body by integration.

Let r = radius of base of a cylinder

h = altitude

γ = density

dV = element of volume

ρ = distance of the element from the axis

Fig. 5.

Then
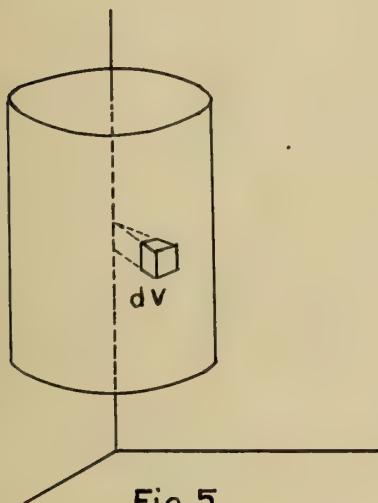
$$I = \gamma \int dV \, \rho^{2}$$

$$d\bar{V} = \rho \, d\theta \, d\rho \, dh$$

$$I = \gamma \int_{0}^{r} \int_{0}^{2\pi} \int_{0}^{h} \rho^{3} \, d\theta \, d\rho \, dh$$

$$= h\gamma \int_{0}^{r} \int_{0}^{2\pi} \rho^{3} \, d\theta \, d\rho = 2\pi h \gamma \int_{0}^{r} \rho^{3} \, d\rho$$

$$= \frac{\pi h r^{4} \gamma}{4} \qquad \text{But } \pi r^{2} h \gamma = M$$

$$\therefore I = \frac{M}{2} r^{2} \qquad k^{2} = \frac{r^{2}}{2}$$

It is evident that these expressions will always be in their simplest form,

when the axis of inertia is the centroid-
al axis of the body. If the moment of
inertia is desired for any parallel axis
it is usually reduced from this centroidal
axis. An expression will now be given
by means of which this reduction may
be effected.
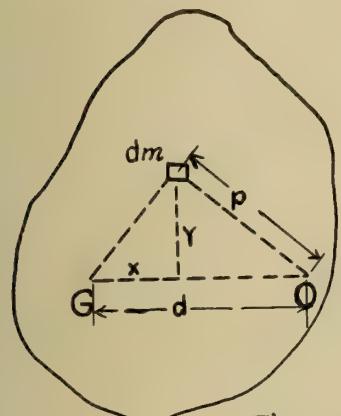
Fig.6.

Let Fig.6 show
section of a body perpen-
dicular to the inertia
axis. Let G and O be
points where the iner-
tia and parallel axes
pierce the plane of
the section.

Let $\overline{I}$ = Moment of inertia with respect to axis G.

I =   "   "   "   "   "   "   " O

d = Distance between O and G.

$$I = \int dM \, \rho^2$$

$$\rho^2 = y^2 + (d-x)^2$$

$$\therefore I = \int dM \left[y^2 + d^2 - 2dx + x^2 \right]$$

$$= \int dM (y^2 + x^2) + \int dM \, d^2 - 2\int dM \, dx$$

But $\int dM (x^2 + y^2) = \overline{I}$

$$\int dM \, d^2 \quad = M d^2$$

$$-2 \int dM \, dx = 0 \qquad (G \text{ is center of gravity.})$$

$$\therefore \quad I = \bar{I} + M d^2$$

Dividing by M, we have

$$\frac{I}{M} = \frac{\bar{I}}{M} + \frac{M d^2}{M} \quad \text{or} \quad k^2 = \bar{k}^2 + d^2$$

These equations are valuable in obtaining the moments of inertia of a composite body, for example, a fly wheel. By integration or by general formulae, the moment of inertia of the rim may be found, considering it as a hollow cylinder. Next the moment of inertia of the arms are found about a centroidal axis and then reduced to the axis of the shaft. Finally, these quantities are all added together and the result is the moment of inertia of the entire wheel.

But, as in the case of finding the center of gravity, the bodies with which we have to deal are for the most part of such shapes that experimental rather than analytical methods must be adopted. There are several ways

in which we may find the moment of
inertia of irregular bodies, but the
two in general use are the pendulum
method and the torsion balance method.

(a) The pendulum method.
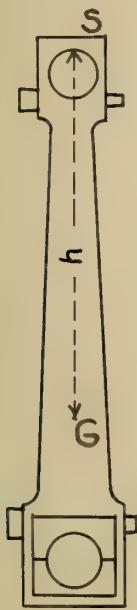
Suspend link as
shown and allow it
to oscillate, observing
carefully the time, T.
The equation of the
pendulum is

$$T = 2\pi \sqrt{\frac{k^2}{hg}} \quad (Maurer's\ Mech.)$$

$$or\ k = \frac{T}{2\pi} \sqrt{hg}$$

Fig. 7.

h = distance from point of
suspension to center of
gravity.

All quantities being known, k can be
readily calculated. The principal radius
of gyration can be found by the equa-
tion already given.

(b) - Torsion Balance.

A flat plate is suspended
by an elastic wire which is firmly
fastened to some support. The body

whose moment of inertia is to be found
is laid upon the plate along a second
body so as to leave the plate in a
horizontal position. The position of the
first is such that the inertia axis
is parallel to the wire, and that of
the second such that its I with
respect to the wire is known. Let
the balance oscillate, and observe the
time, T; remove the bodies and let
the empty balance oscillate again
and observe the time, T'.

Let $I_1 = $ I of plate with respect to wire
  $I_2 = $ " " second body "   "   "   "
  $I = $ " " link   "   "   "   ".

From equation of balance

$$T = \pi \sqrt{\frac{I}{c}} \qquad \text{(Maurer's Mechanics)}$$

we have that the I's are proportional
to the squares of the time or

$$T : T' :: \sqrt{I_1 + I_2 + I} : \sqrt{I}$$

or $\quad I = \dfrac{I_1 T^2}{T'^2} - [I_1 + I_2]$

Concentrated Mass - When a body
is considered to be concentrated at one

point, this consideration implies that it
has no volume. Therefore, its moment of
inertia with respect to an axis through
the point is zero, and with respect to
any other axis is simply Md^2 when M
is the mass and d the distance be-
tween the axis and the point where the
mass is considered to be concentrated.

3 - <u>Kinetic Energy</u> - A body possess-
es energy when it is capable of doing
work in changing its state. Energy
which is due to velocity of body is
known as kinetic energy. As work
can be directly converted into energy
and energy into work, these quantities
must be expressed in the same units,
which in the Engineer's System are
foot-pounds.

The fundamental equation for
kinetic energy is derived from the re-
lation that energy is equal to work
done.

Work is product of force and dis-
tance.

$$W = \int_0^s F \cdot ds = E_K$$

But $\qquad F = \Delta M a = \Delta M \dfrac{dv}{dt}$

$$E_K = \int_0^v \Delta M \frac{dv}{dt} ds = \int_0^v \Delta M \cdot v \, dv$$

$$= \frac{1}{2} \Delta M v^2$$

A body may be considered as made up of a system of particles. For translation, the velocity of all particles is the same. Then

$$E_K = \Sigma \frac{1}{2} \Delta M v^2$$
$$= \frac{1}{2} M v^2$$

For a rotating system, the velocity of any particle is $r\omega$. Then

$$E_K = \Sigma \frac{1}{2} \Delta M (r\omega)^2$$
$$= \frac{\omega^2}{2} \Sigma \Delta M r^2$$
$$= \frac{1}{2} I' \omega^2$$

This equation has a wider application than the first for it may be applied to a body with any plane motion, for a body may be considered

as rotating about its instantaneous center. In this case the moment of inertia would in general be constantly varying as the instantaneous center of rotation is not fixed. The kinetic energy may then be divided into two parts, one due to a rotation about the center of gravity, and the second due to the translation of the center of gravity itself. We have therefore

$$I' = \bar{I} + M r^2$$

$$E_K = \frac{1}{2} I' \omega^2 = \frac{1}{2} \omega^2 \left[I + M r^2 \right]$$

$$= \frac{1}{2} \omega^2 I + \frac{1}{2} M v^2$$

4 - <u>Equations</u> <u>for</u> <u>Reducing</u> <u>Masses</u> - General expressions which can be used for reducing the mass of a body to any point will now be deduced. The following notation will be used in all subsequent work. Original masses will be denoted by roman capitals with suitable subscripts, reduced masses with corresponding script capitals.

The first case to be considered

is the reduction of a rotating mass
to any point in the same plane of
rotation. The principle for reducing
masses has already been enunciated,
namely, that the kinetic energy of the
reduced mass at any instant must e-
qual the total energy of all separate
masses moving at that instant.

Let O = center of rotation

P = point to which
mass is to be
reduced

G = center of gravity

r = distance from O to P

Take any differential
element of mass, d M,

Fig. 8.

at a distance x from O. Let d M be cor-
responding element reduced to point P.
The kinetic energy of a particle. d M

$$= \frac{1}{2} \, dM \, x^2 \left(\frac{d\theta}{dt}\right)^2$$

The kinetic energy of particle d M

$$= \frac{1}{2} \, dM \, r^2 \left(\frac{d\theta}{dt}\right)^2$$

Hence $\frac{1}{2} \, dM \, x^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2} \, dM \, r^2 \left(\frac{d\theta}{dt}\right)^2$
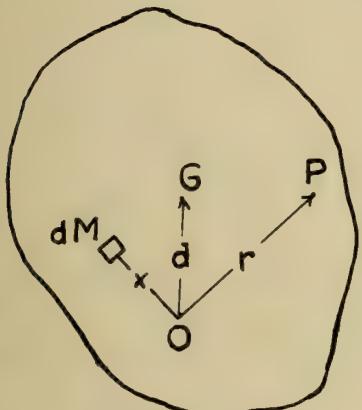
$$\int dM x^2 = r^2 \int dM$$

But $\int dM x^2 = I'$

therefore $I' = M r^2$

or $M = \dfrac{I'}{r^2} = \dfrac{I + Md^2}{r^2}$

This equation stated in words is as follows: _The reduced mass at any point is equal to the moment of inertia divided by the square of the distance from the center of rotation to that point._ The equation may be used for solving various problems relating to rotating bodies, and in many cases the solution is thus considerably simplified. A few examples will serve to illustrate the application.
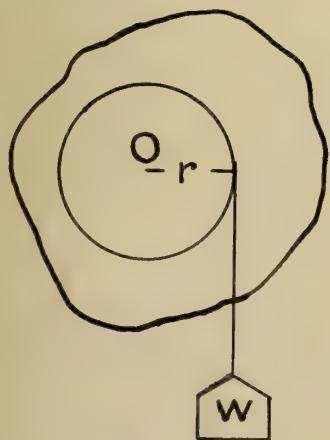
Fig. 9.

The body shown in figure 9 is made to rotate by a weight W, which is fastened to a cord wrapped about a cylindrical portion of the body
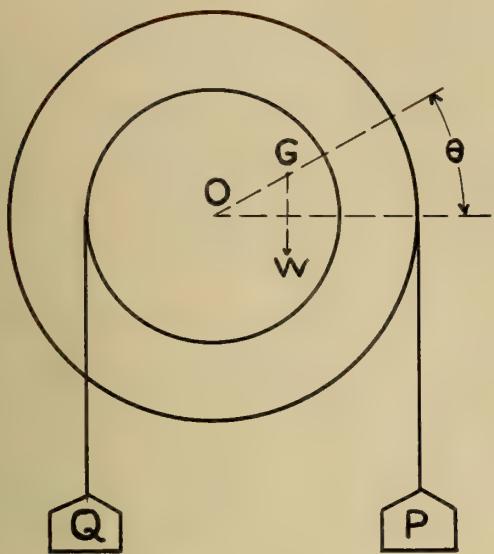
Fig. 10.

In figure 10, let

W = weight of rotating body.

$\bar{r}$ = radius of center of gravity

r = radius of Q

R = radius of P

a = acceleration of P

W reduced to radius of

$$P = \frac{I'}{R^2}.$$

Total mass moved $= \dfrac{I'}{R^2} + \dfrac{P}{g} + \dfrac{Qr}{Rg}$

Forces acting are P, Q and $W\bar{r}\cos\theta$, therefore the total force at radius R

$$= P + \frac{W\bar{r}\cos\theta}{R} - \frac{Qr}{R}$$

We now have

$$P + \frac{W\bar{r}\cos\theta}{R} - \frac{Qr}{R} = \left[\frac{I'}{R^2} + \frac{P}{g} + \frac{Qr}{Rg}\right]a$$

or

$$a = \frac{PR + W\bar{r}\cos\theta - Qr}{\left[\dfrac{I'}{R} + \dfrac{PR}{g} + \dfrac{Qr}{g}\right]}$$

The acceleration for any given angle of θ can now be computed.
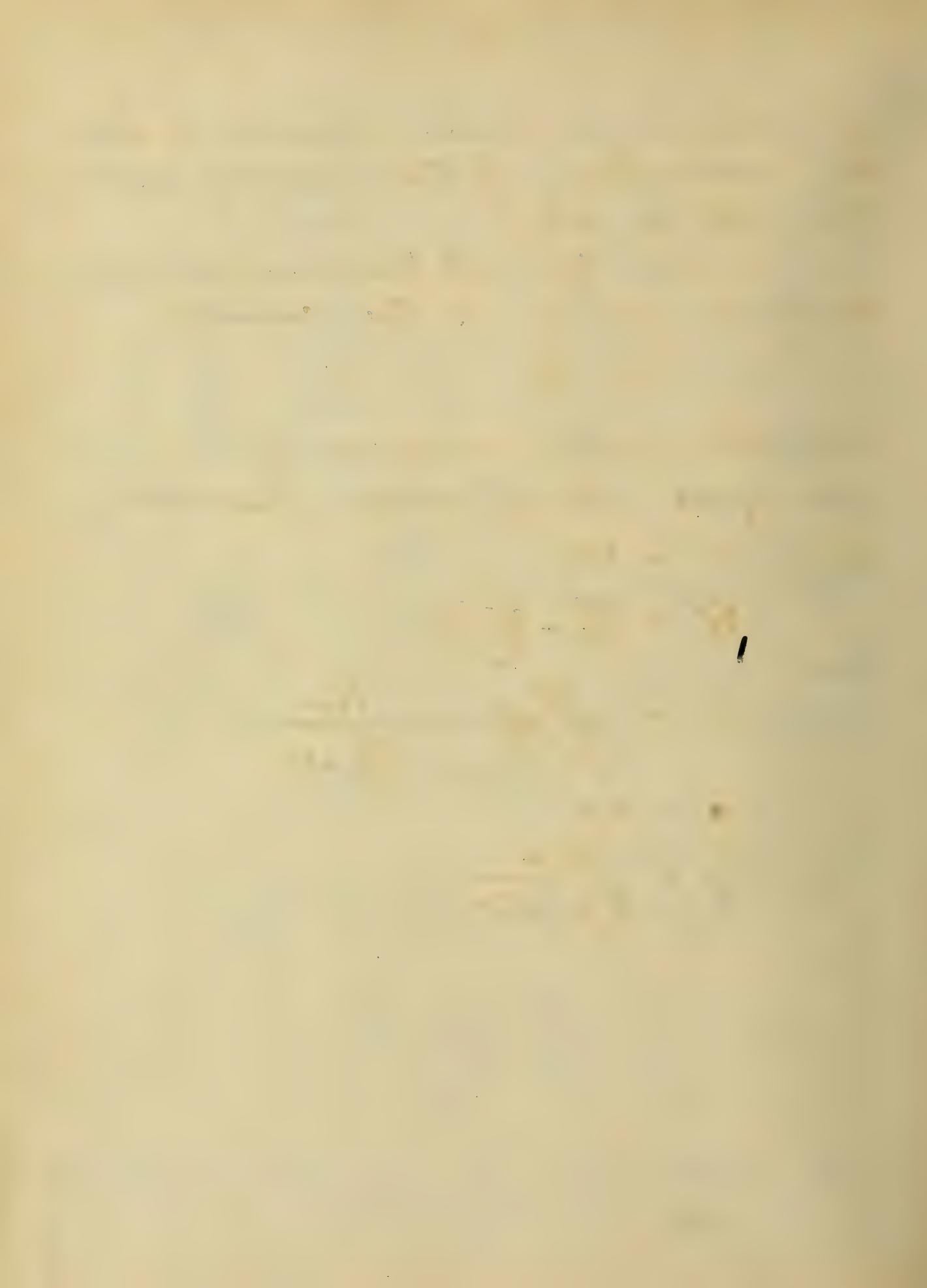
of radius r. It is required to find the acceleration of the weight and of the rotating body.

Reduce the rotating mass to a point on surface of the cylinder.

$$M = \frac{I'}{r^2}$$

The total mass moved is $\frac{I'}{r^2} + \frac{W}{g}$

The force acting is the weight W

But $F = Ma$

$$\therefore \quad W = \left[\frac{I'}{r^2} + \frac{W}{g}\right] a$$

$$a = \frac{W}{\left[\frac{I'}{r^2} + \frac{W}{g}\right]} = \frac{Wg}{\frac{I'g}{r^2} + W}$$

$$a = r\alpha$$

$$\therefore \quad \alpha = \frac{Wg}{\frac{I'g}{r} + Wr}$$

The advantage of using reduced masses in solution of the preceding problems is now evident. The problems can be solved at once from one equation. If the method of moments is used, the acceleration is found by solving simultaneous equations with three or more variables.

The second case is the reduction of a mass having a motion of translation to a point of a rotating system.
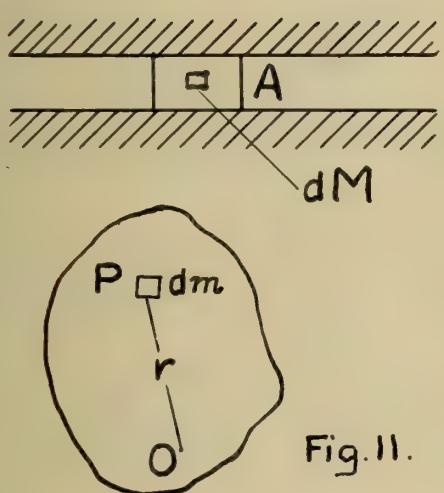
In figure 11, let
A = translating system
B = rotating system
O = center of rotation
P = point to which A is to be reduced

Fig. 11.

Kinetic energy of particle, $dM = \frac{1}{2} dM \left(\frac{dx}{dt}\right)^2$

Kinetic energy of particle, $dM = \frac{1}{2} dM r^2 \left(\frac{d\theta}{dt}\right)^2$

$\therefore \frac{1}{2} dM \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} dM r^2 \left(\frac{d\theta}{dt}\right)^2$

or $\qquad M = \frac{M}{r^2} \frac{\left(\frac{dx}{dt}\right)^2}{\left(\frac{d\theta}{dt}\right)^2}$

Before this expression can be determined the relation between $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ must be known. This relation is given by the geometry of the figure. Take, for example, the familiar slider crank mechanism shown in figure.

Fig. 12

Let r = radius of crank.

l = length of connecting rod.

θ = angle of crank.

It is required to reduce the reciprocating mass M to the radius of the crank. From preceding equation it is evident that the velocity of the crosshead must be found.

From figure

$$OS = X = r + l - r\cos\theta - \sqrt{l^2 - r^2\sin^2\theta}$$

$$\frac{dx}{dt} = r\sin\theta\frac{d\theta}{dt} + \frac{r^2\sin\theta\cos\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\cdot\frac{d\theta}{dt}$$

But $\dfrac{d\theta}{dt} = \omega$ the angular velocity of the crank.

Therefore $\dfrac{dx}{dt} = r\omega\left[\sin\theta + \dfrac{\sin\theta\cos\theta}{\left(\dfrac{l^2}{r^2} - \sin^2\theta\right)^{\frac{1}{2}}}\right]$

Now $m = \dfrac{M}{r^2\omega^2}\left(\dfrac{dl}{dt}\right)^2$

Therefore $m = M\left[\sin\theta + \dfrac{\sin\theta\cos\theta}{\left(\dfrac{l^2}{r^2} - \sin^2\theta\right)^{\frac{1}{2}}}\right]^2$

This expression is often simplified by the introduction of approx-imations which are allowable when the ratio of the connecting rod to the crank is large. This condition is obtained when the mass of the valve and valve rod is reduced to the radius of the eccentric.
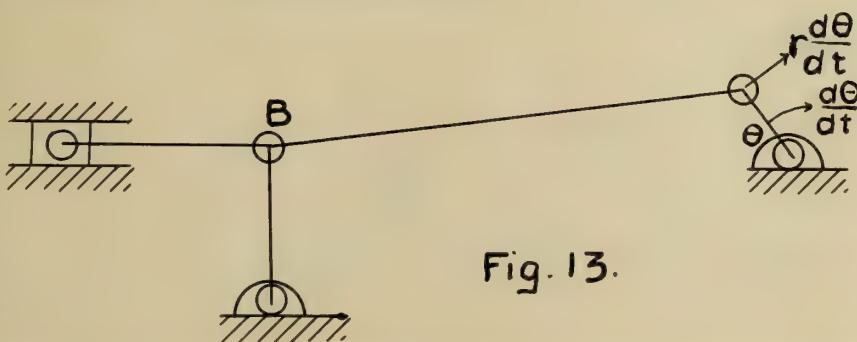
Fig. 13.

Linear velocity of $A = r\dfrac{d\theta}{dt}$

Velocity of B is approximately

$r\dfrac{d\theta}{dt}\sin\theta$

Therefore $\mathcal{M} = \dfrac{M\left(\dfrac{dx}{dt}\right)^2}{r^2\left(\dfrac{d\theta}{dt}\right)^2} = \dfrac{Mr^2\left(\dfrac{d\theta}{dt}\right)^2\sin^2\theta}{r^2\left(\dfrac{d\theta}{dt}\right)^2}$

$\mathcal{M} = M\sin^2\theta.$

The last case to be considered is the reduction of the mass of any link of a machine to some point on another link.

I_{in} = moment of inertia with respect to its instantaneous center

in = instantaneous center of link $\underline{n}$.

ω_{in} = angular velocity about instantaneous center.

ip = instantaneous center of $\underline{p}$.

E = point on link p to which $\underline{n}$ is to be reduced.

r = radius of E from ip.

ω_{pi} = angular velocity of link p.

$$I'_{in} = I + M\left[in - G_{in}\right]^2$$

Kinetic energy of link $n = \frac{1}{2} I'_{in} \omega_{in}^2$

Hence kinetic energy of reduced mass $= \frac{1}{2} M r^2 \omega_{p_i}^2$

$$\frac{1}{2} M r^2 \omega_{pi}^2 = \frac{1}{2} I'_{in} \omega_{in}^2 \qquad or$$

$$M = \frac{I'_{in} \omega_{in}^2}{\omega_{pi}^2 r^2} = \frac{M\left[K^2 + (in - G_{in})^2\right]}{r^2} \frac{\omega_{in}^2}{\omega_{pi}^2}.$$

To illustrate the use of these expressions just derived we will take an existing flyball governor and reduce all masses to the center of the ball. Two conditions should be carefully noted. The motion of the governor about its own axis is not con- sidered. The fly weight is given a slight movement and the kinetic energy of all of the links is that

due to the velocity which
is obtained by this motion.
It should also be observed
that for every expression
derived for reduced mass
one of the factors always
is the ratio of the velocities.
This shows that the rate
which the mechanism is
moving does not affect
the result, because the ratio
of the velocities of the two
links depends only upon
the geometry of the figure.

Figure 14 is a sketch
of the governor on the
York Ice Machine in the Mech-
anical Engineering Laboratory

Let I = moment of inertia
of link ABP with
respect to center of
rotation.
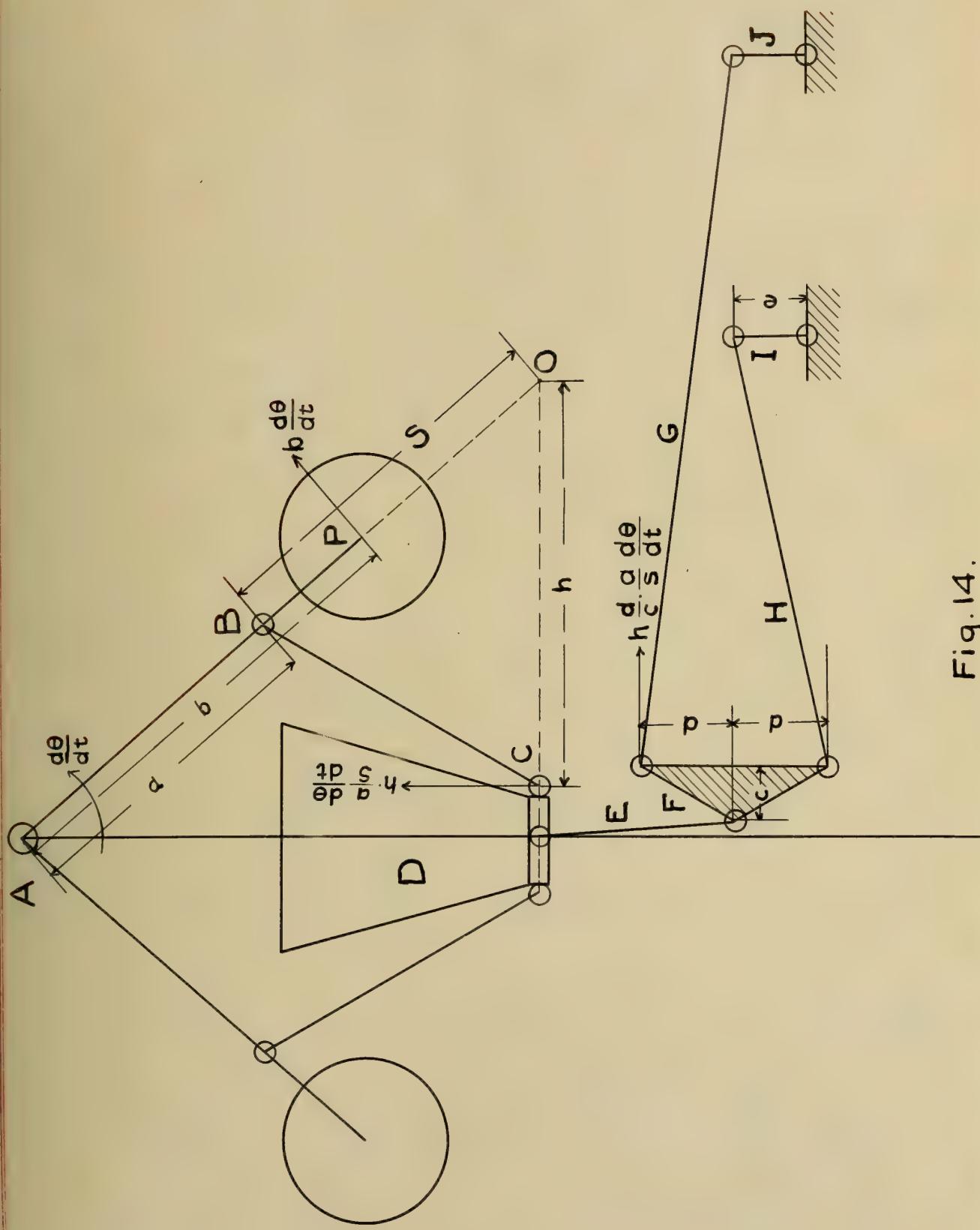
Fig. 14.

I_1 = moment of inertia of link CB with respect to instantaneous center of rotation.

M_1 = mass of D.

M_2 = mass of E.

I_2 = moment of inertia of F with respect to center of rotation.

M_3 = mass of G.

M_4 = mass of H.

I_3 = moment of inertia of I and J with respect to center of rotation.

The governor has an angular velocity $\dfrac{d\theta}{dt}$ about the pin A

Linear velocity of $P = b\,\dfrac{d\theta}{dt}$.

Kinetic energy of $ABP = \frac{1}{2}\, I \left(\dfrac{d\theta}{dt}\right)^2$

Kinetic energy of $d\,M = dm\,\dfrac{b^2}{2}\left(\dfrac{d\theta}{dt}\right)^2$

Therefore $\frac{1}{2}\, I\left(\dfrac{d\theta}{dt}\right)^2 = \sum dm\,\dfrac{b^2}{2}\left(\dfrac{d\theta}{dt}\right)^2$

Or $M = \dfrac{I}{b^2}$ = mass of link ABP reduced to P.

Link CB rotates about instantaneous center O. Linear velocity of B is $a\frac{d\theta}{dt}$, therefore angular velocity about O is $\frac{a}{s}\frac{d\theta}{dt}$.

Kinetic energy $= \frac{1}{2} I_1 \left(\frac{a}{s} \frac{d\theta}{dt} \right)^2$

And as before

$$\frac{1}{2} I_1 \left(\frac{a}{s} \frac{d\theta}{dt} \right)^2 = M_2 \frac{b^2}{2} \left(\frac{d\theta}{dt} \right)^2$$

$$M_2 = \frac{I_1 a^2}{s^2 b^2}$$

Point C has linear velocity $h\frac{a}{s}\frac{d\theta}{dt}$ which is also velocity of D and a close approximation to that of E. We shall therefore consider D and E as one body. We have therefore directly

$$M_3 = \frac{M_1 + M_2 \left(h\frac{a}{s} \frac{d\theta}{dt} \right)^2}{b^2 \left(\frac{d\theta}{dt} \right)^2} = M_1 + M_2 \left(\frac{h}{b} \cdot \frac{a}{s} \right)^2$$

The angular velocity of F is $\frac{h}{c} \cdot \frac{a}{s} \cdot \frac{d\theta}{dt}$ and its kinetic energy is

$$\frac{I_2}{2} \left[\frac{h}{c} \cdot \frac{a}{s} \cdot \frac{d\theta}{dt} \right]^2 \quad or$$

$$\frac{I_2}{2}\left[\frac{h}{c} \cdot \frac{a}{5} \cdot \frac{d\theta}{dt}\right]^2 = M_4 \frac{b^2}{2}\left(\frac{d\theta}{dt}\right)^2$$

$$M_4 = I_2\left[\frac{1}{b} \cdot \frac{h}{c} \cdot \frac{a}{5}\right]^2$$

The motion of the rods G and H may be taken as a translation. The velocity is $d \cdot \frac{h}{c} \cdot \frac{a}{5} \cdot \frac{d\theta}{dt}$

Therefore $M_5 = \dfrac{M_3\left[d \cdot \frac{h}{c} \cdot \frac{a}{5} \cdot \frac{d\theta}{dt}\right]^2}{\left[b \cdot \frac{d\theta}{dt}\right]^2}$

$$M_5 = M_3\left[\frac{d}{b} \cdot \frac{h}{c} \cdot \frac{a}{5}\right]^2$$

In the same way

$$M_6 = M_4\left[\frac{d}{b} \cdot \frac{h}{c} \cdot \frac{a}{5}\right]^2$$

Angular velocity of link 1 is $\dfrac{d}{l} \cdot \dfrac{h}{c} \cdot \dfrac{a}{5} \cdot \dfrac{d\theta}{dt}$ and its kinetic energy is

$$\frac{I_3}{2}\left[\frac{d}{l} \cdot \frac{h}{c} \cdot \frac{a}{5} \cdot \frac{d\theta}{dt}\right]^2$$

Therefore

$$\frac{M_7}{2} b\left(\frac{d\theta}{dt}\right)^2 = \frac{I_3}{2}\left[\frac{d}{l} \cdot \frac{h}{c} \cdot \frac{a}{5} \cdot \frac{d\theta}{dt}\right]^2$$

$$M_7 = I_3 \left[\frac{1}{b} \cdot \frac{d}{l} \cdot \frac{h}{c} \cdot \frac{a}{5} \right]^2$$

As I and J are of the same size and have the same velocity the last expression also gives the mass of J reduced to P.

The addition of the quantities just obtained gives the reduced mass of all links of the governor that accumulate kinetic energy. It is at once evident that this sum varies for each configuration of the mechanism. If this reduced mass were calculated for several different positions and a perpindicular erected upon a straight line base directly below the point P and equal to the quantity thus obtained a smooth curve through these points would give the reduced mass curve. When this curve is once obtained the reduced mass for any position of the governor

can be scaled from the curve.

In the preceeding discussion analytical methods were used entirely simply for the sake of clearness. For complicated mechanisms however, the expressions for velocities become so long and involved that it is necessary to substitute graphical methods. This may be done by means of instantaneous centers or velocity polygons. The latter method gives an elegant solution because when a velocity polygon has been constructed it gives the velocity of every point of the mechanism.

III – Force Reduction.

1. <u>Forces</u> <u>arising</u> <u>which</u> <u>may</u> <u>be</u> <u>reduced</u>.

Whenever a mechanism changes its configuration, energy is imparted to it by forces arising from the accelerations of the moving masses. In order to effect a reduction of these forces to some point the following condition must hold. The energy imparted to the reduced system must be the same as that imparted to the actual system. In other words, the work of the reduced force for any interval of time must be equal to the sum of the works performed by the actual driving forces for the same interval.

Force is usually defined as the action of one body upon another that tends to change motion, either in direction or magnitude. In fact the term

force was introduced to account
for the observed changes of motion
of bodies. Our first conception
of force is founded on our ex-
perience with forces exerted by
ourselves.

The first force to be con-
sidered acting upon the links
of a machine is the force of
gravity or simply the weight.
The reduction of this force should
not be confused with the reduct-
ions of the masses, given in a
previous paragraph. The remaining
forces which arise are due to
the accelerations of the moving
masses. Let us first consider
the simplest case, namely, that of
a body rotating about a fixed
center.

Fig. 15
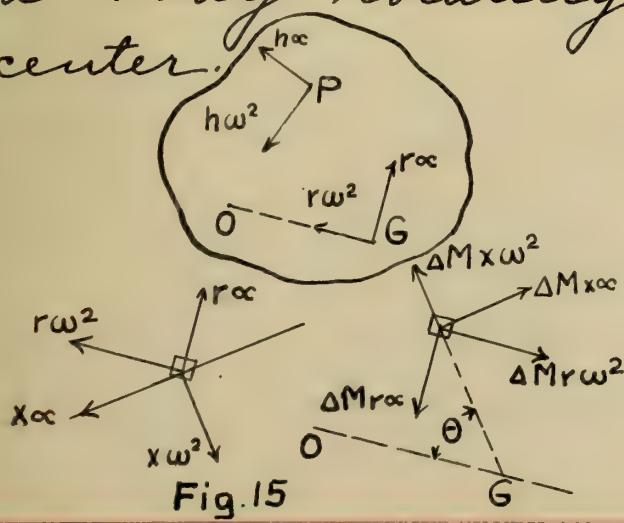
In figure 15 let
P = any point
G = the center of
gravity.
O = the center of
rotation.
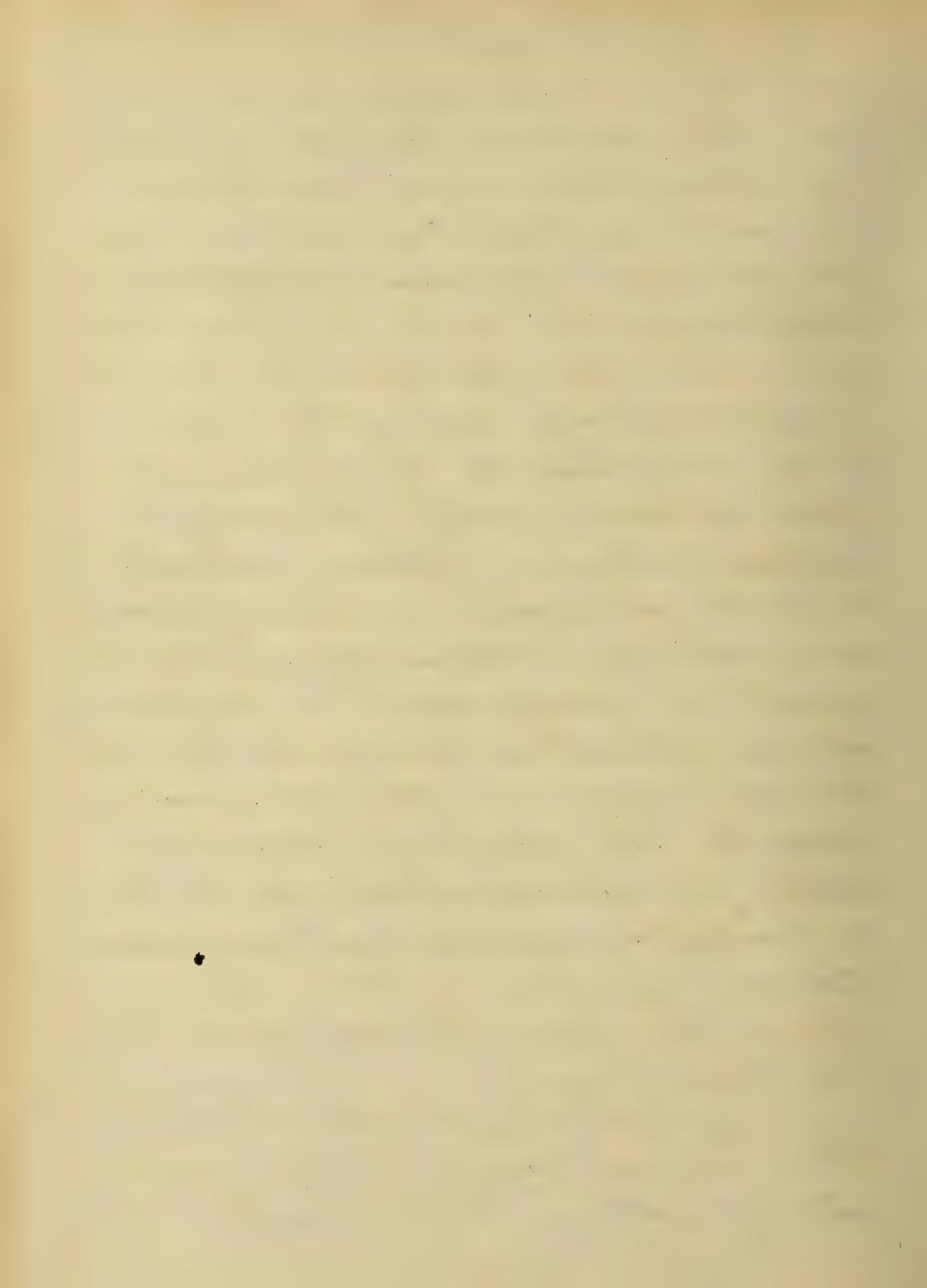
r = the radius of G

h = the distance from O to P.

x = the distance from G to P.

The acceleration at P may be divided into two components, the tangential and the normal. The value of the former is hx and that of the latter $h\omega^2$. For convenience of summation, the rotation about O may be replaced by a rotation about G with an equal angular velocity ω, and a translation which must be perpindicular to OG. The acceleration components of P will now be the acceleration due to the rotation about G and the acceleration due to translation which is common to all points of the body. These components are also shown in the figure.

Let ΔM denote an element of mass at point P; the accelerating forces at P are found

by multiplying the acceleration
components by ΔM. By D'Alemberts
principal these forces reversed
in sense will hold the external
forces acting on ΔM in equi-
librium. The reversed forces
are shown in the figure. We
will now sum each of these
forces for the entire body.

$$\Sigma \, r\omega^2 \Delta M = M r \omega^2 = F_r$$

The forces $\Delta M r \omega^2$ form a system
of parallel and equal forces,
therefore they have a single re-
sultant which passes through
the center of gravity. The forces
$r \alpha \Delta M$ are all equal and per-
pendicular to OG. They there-
fore have the single resultant.

$$\Sigma \, r \alpha \Delta M = M r \alpha = F_t.$$

To find the resultant of forces

$x\omega^2 \Delta M$ resolve them into components parallel and perpendicular to OG.

Then,

$$\Sigma \, x\omega^2 \Delta M \cos \theta = 0$$
$$\Sigma \, x\omega^2 \Delta M \sin \theta = 0$$

The resultant of this system therefore reduces to zero.

There now remains the forces $x\alpha\Delta M$. The moment of the force $x\alpha\Delta M$ about G is $x^2\alpha\Delta M$. The summation gives as the total moment,

$$M = \Sigma \, x^2\alpha \, \Delta M = \alpha \, \Sigma \, x^2\Delta M = I\alpha.$$

Therefore in a rotating body we have two forces F_t and F_r which act through the center of gravity and a moment $I\alpha$. F_r is commonly known as the centrifugal force and sometimes as the radial inertia force. F_t is called the tangential inertia

force. The couple $I\alpha$ may be considered as due to the masses resistance to angular acceleration about its own center of gravity.

The total moment about the center O is the sum of the moment of the couple $I\alpha$ and the moment of F_t. F_r has no moment about O.

$$M_{total} = I\alpha + Mr\alpha \cdot r = \alpha \left[I + Mr^2 \right]$$

$$= I'\alpha$$

Therefore to give the body an angular acceleration about O an external moment of $I'\alpha$ is required.
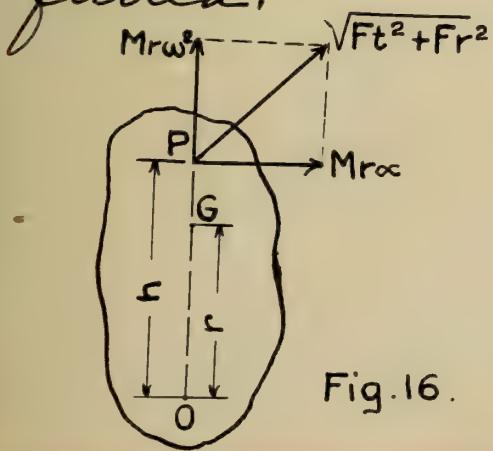
Fig. 16.

F_t, F_r, and $I\alpha$ may be replaced by a single force whose magnitude is $\sqrt{F_r^2 + F_t^2}$, see figure 16, and

which acts through a point on the radius drawn through the center of gravity and at a distance $k^2/\bar{r}$. This may be proven as follows:

The total moment about O is $I'\alpha$
Moment of single force is $M\bar{r}\alpha h$
Therefore,

$$M\bar{r}\alpha h = I'\alpha$$

Whence,

$$h = \frac{I'}{M\bar{r}} = \frac{Mk^2}{M\bar{r}} = \frac{k^2}{\bar{r}}.$$

The expressions just deduced may be applied to a body rotating about an instantaneous center.
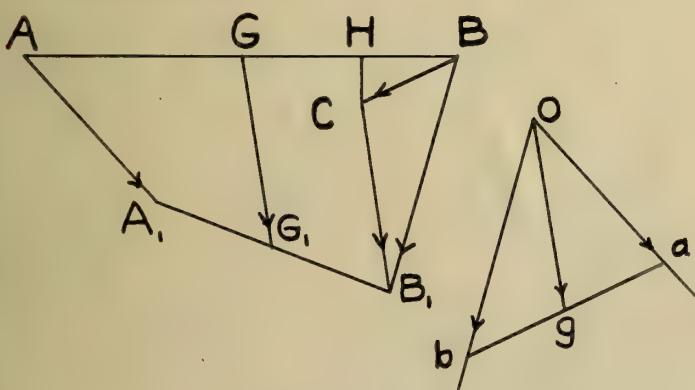
Fig.17.

In figure 17 let AB be the link of a machine rotating about some instantaneous center. Let AA_1 and BB_1 be accelerations of A and

B respectively. From the acceler-
ation polygon the magnitude and
direction of the acceleration of
the center of gravity may be found.
We may replace the motion of AB
by a rotation about G and a
translation perpendicular to the
line joining G and the instan-
taneous axis. The acceleration of
any point will now be made
up of two components, one due
to the rotation about G and the
other due to translation. G's ro-
tational components are zero there-
fore G's acceleration is due to
translation and is common to
all points of the body. Resolve
BB_1 into components BC and
$CB_1 (= GG_1)$, BC is the rotation
component. If BC is again re-
solved into components perpen-
dicular and parallel to AB the
tangential acceleration of B, HC_1
may be obtained.
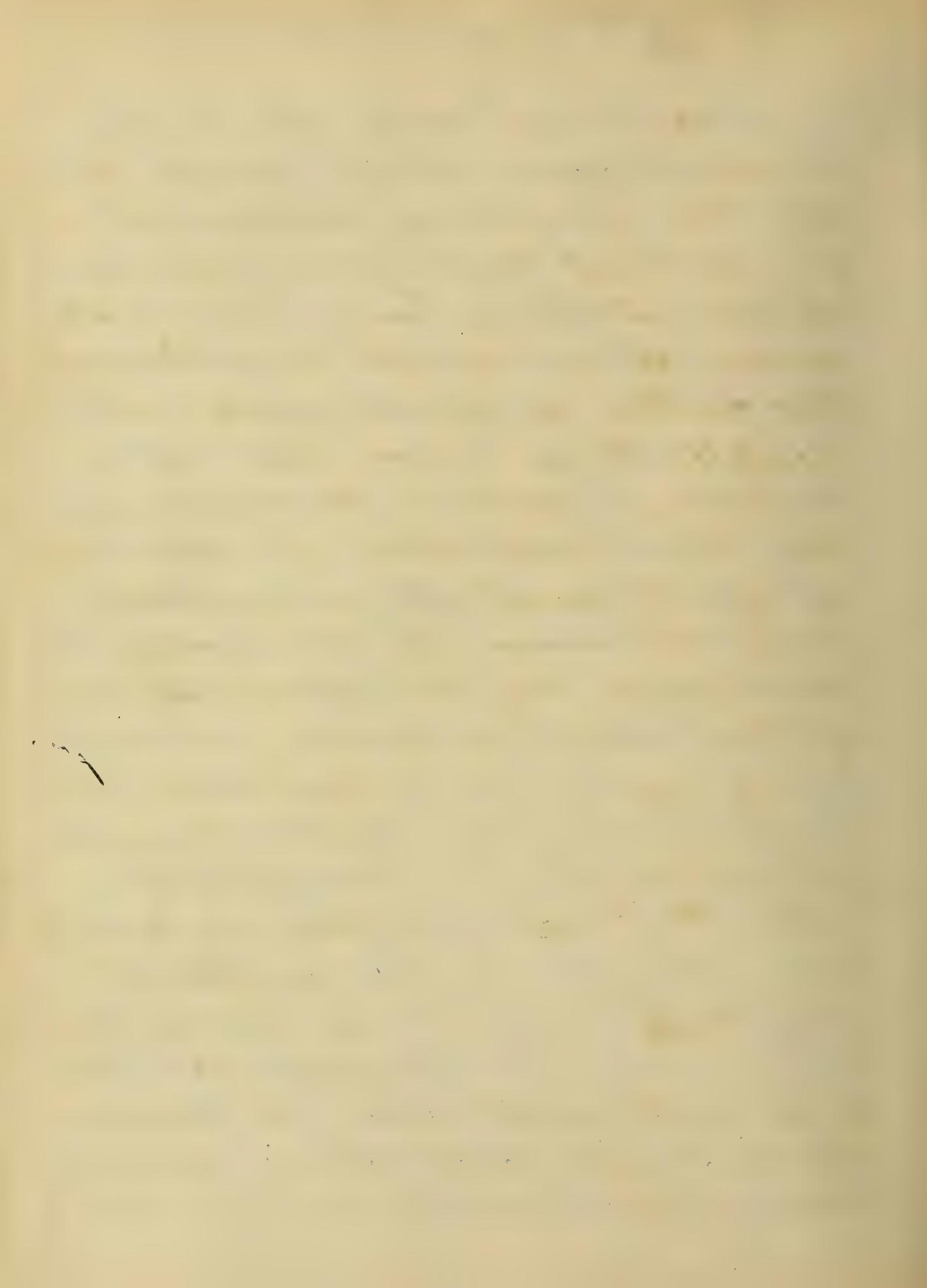
We have then $\frac{HC}{GB} = \alpha =$ angular acceleration about G. In the preceding paragraph we found that the inertia forces of a rotating body were composed of a single force through the center of gravity and a couple. This force had a direction opposite to that of the total acceleration of the center of gravity and its magnitude was the mass of the body multiplied by the acceleration of the center of gravity. In this case we have therefore a force Q through G and a couple $I\alpha$ as shown in figure 18. The
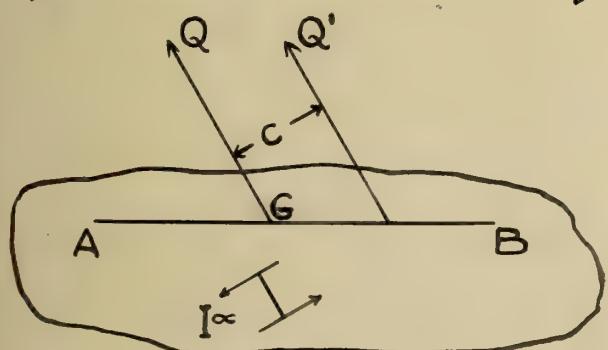
Fig. 18

force and couple may be replaced by a single force whose action line is parallel to Q but does

not pass through G. Then a point
at a distance c from Q intro-
duce a force Q' equal and par-
allel to Q. The moment of Q'
about G is Q'c.

$$\text{But} \quad Q'c = I\alpha$$
$$\text{Therefore} \quad c = \frac{I\alpha}{Q'}$$

This equation gives the distance
through which the action line
must be shifted from the center
of gravity in order to produce
the given acceleration.

Q' = mass times acceleration of
G and α can be found by
method already shown. All terms
being known c can be read-
ily calculated.

Another method of finding the
action line of the inertia force is
to replace the body under consid-
eration by a kinetically equivalent
system. Two systems are kinetically
equivalent when to give them the

same motion equal translation
forces and equal rotation forces
are required. The conditions that
must be fullfilled therefore are,
first, systems must have same
mass; second, the same center of
gravity, and third, the same mom-
ent of inertia. Let origional body
have a mass of m gee pounds
and the equivalent system be
composed of two masses m_1 and
m_2 concentrated at the proper
points. Applying above condition
we have,

1. $m_1 + m_2 = m$ Systems have same
mass.

2. $m_1 h_1 = m_2 h_2$ Systems have same
center of gravity.

3. $m_1 h_1^2 + m_2 h_2^2 = m k^2$ Systems have
same moment of
inertia.

Eliminating m_1, m_2, and m from
these three equations:

$$k^2 = h_1 h_2$$

K is the mean proportional be-
tween h_1 and h_2. Hence if the
position of one mass and K be
given, the position of the second
mass can be found by the
following graphical construction.
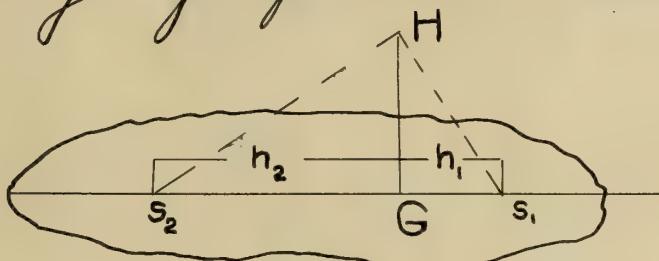
Fig. 19.

In figure 19 let G be the mass
center of a body which is sym-
metrical about the lines $S_1 S_2$. Let
S_1 be the given position of one
mass. Draw GH perpendicular
to $S_1 S_2$ and equal to K. Join
H and S_1 and draw $H S_2$ at
right angles to $H S_1$. S_2 is the
position of the second mass.
      When the position of the
two masses has been located
the direction of their acceleration
is found. In figure 20 let AB
represent some body whose center
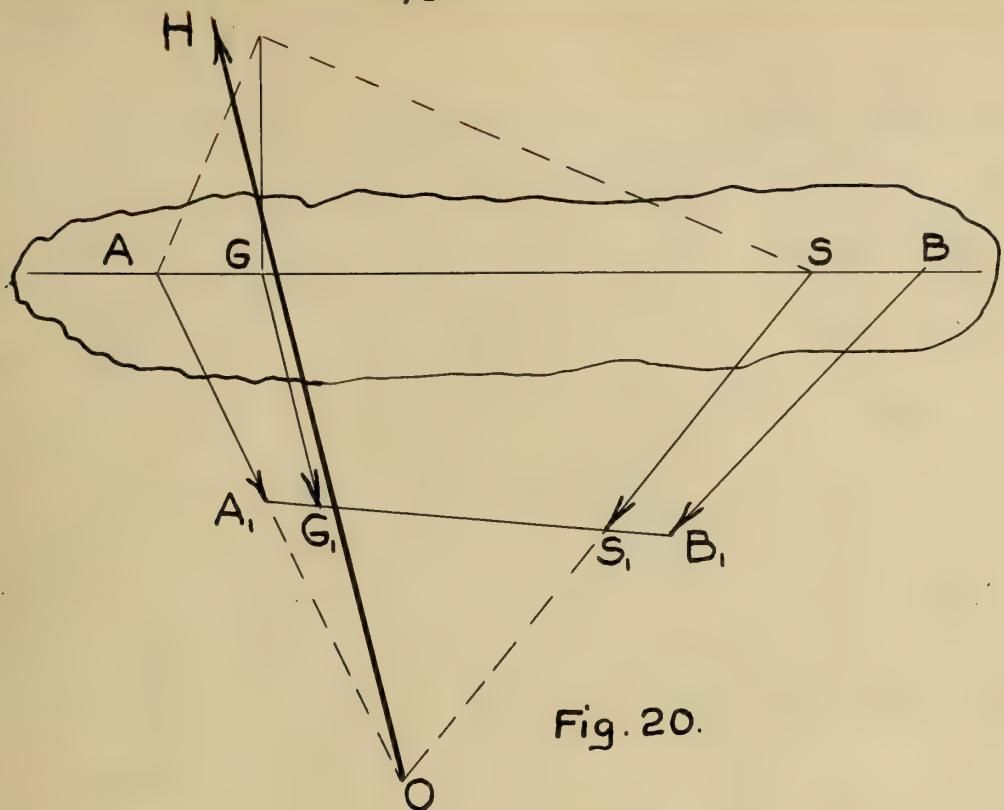
Fig. 20.

of gravity is at G and the acceler-
ation of A and B respectively is
AA, and BB, respectively. The acceler-
ation of any other point as G may
now be found, by laying off A,G,
proportional to AG, Let the first
mass be concentrated at A, then
by given construction find S
the location of the second mass.
Produce the action lines of the
accelerating forces of the two
masses until they intersect at
O. As the inertia force of the

body is the opposite of the resultant accelerating force of the two concentrated masses its action line must pass through O. It has already been shown that the inertia force is parallel to the acceleration of the center of gravity. Therefore through O draw OH parallel to GG, this is the action line required.

A still more general case is that in which a point moved in a path which has some plane motion not a translation. An illustration is a point on the pendulum of a shaft governor. This moves in a circle about the center of suspension and the whole circle has a motion about the engine shaft. In this instance the acceleration of the point is not the resultant of the acceleration of the path, and the point in the path, but is instead the resultant of

these two plus a third component which we shall proceed to find.
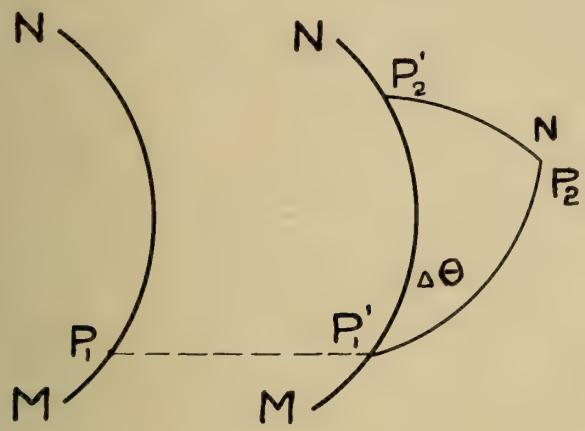
In figure 21 the motion of point P from p_1 to its final position p_2 is made in three steps. First a translation of path MN, then motion in the path $P_1'P_2'$, then a rotation of the path through an angle $\Delta\theta$.

Fig. 21.

Let ΔS = length of path traversed by P.

a = acceleration due to rotation of path.

We have then

$$p_2'p_2 = \frac{1}{2} a \Delta t^2 = \Delta S \cdot \Delta\theta$$

$$a = 2 \frac{\Delta S}{\Delta t} \cdot \frac{\Delta\theta}{\Delta t} \quad \text{approximately.}$$

In the limit therefore

$$a = 2 \frac{ds}{dt} \cdot \frac{d\theta}{dt}.$$
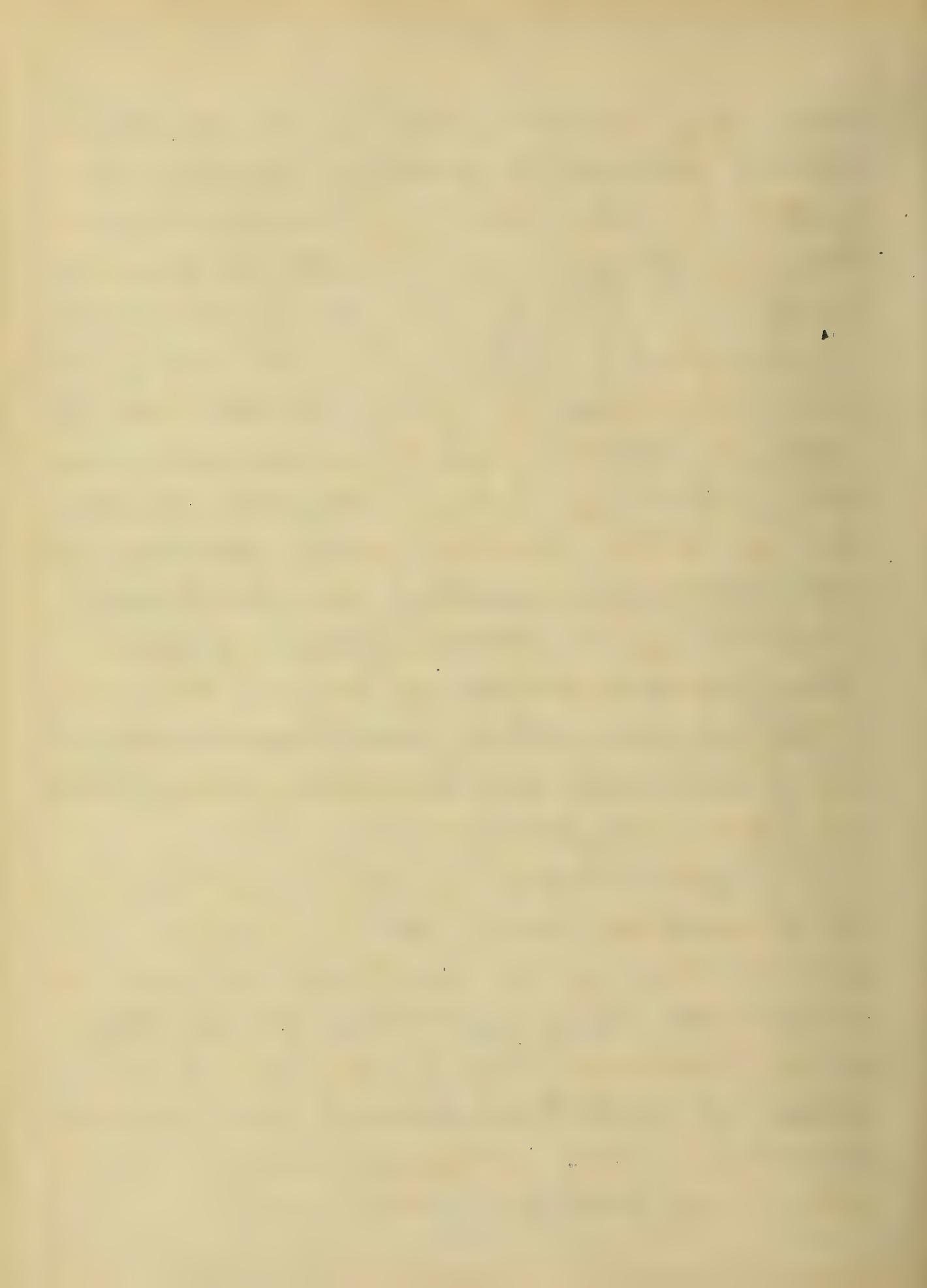
Now $\frac{ds}{dt}$ is the velocity of the point along the curve and is usually de-noted by u and $\frac{d\theta}{dt}$ is the angular velocity of the curve and is denoted by ω

Therefore $a = 2u\omega$

The direction of this third com-ponent is the limiting direction of $P_2 P$, which is normal to the curve or in other words perpendicular to the relative velocity. The total accel-eration of P = acceleration of path plus acceleration of P in the path plus $2u\omega$. This last acceleration is known as the supplementary acc-eleration component.

The most general application of Coriolis' law is to motion in space. In plane motion, it will be remembered, the motion of the path was replaced by a translation and a rotation about an instan-taneous axis perpendicular to the plane of motion. For motion in

space the instantaneous axis is us-
ually inclined at an angle θ with
the path. This introduces another
factor in the expression for accelera-
tion. To make this application clear
a particular example will be taken
up. Consider the fly ball governor
shown in figure 22.

If the gov-
ernor changes its
configuration, the
masses will have
an acceleration
in the plane of
the governor's
motion and also
an acceleration
in a perpendicular
plane, about the
axis AB. Divide
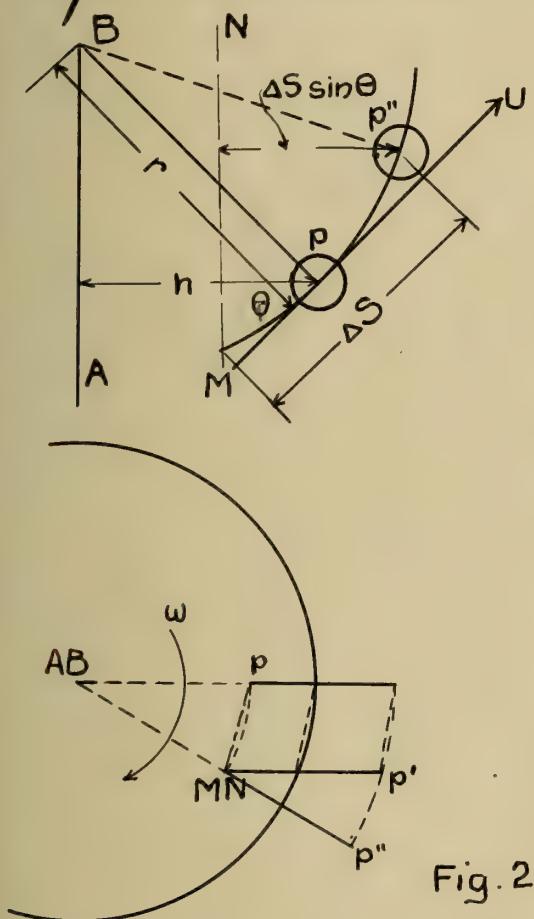
Fig. 22

the motion of point P from p to p"
into a rotation about B and a trans-
lation to p' and a rotation about
the instantaneous axis MN from p' to p"

We now have:

Acceleration of P = acceleration along the path plus acceleration of path plus supplementary acceleration. Supplementary acceleration component = component due to rotation from p' to p''

$$p'p'' = \frac{1}{2} a \Delta t^2 = \Delta \varphi \times radius$$

But radius $= \Delta s \sin \theta$

Therefore

$$\frac{1}{2} a \Delta t^2 = \Delta \varphi \Delta s \sin \theta \quad approximately.$$

In the limit, therefore

$$a = 2 \frac{ds}{dt} \cdot \frac{d\theta}{dt} \sin \theta$$

$$= 2 u w \sin \theta$$

We now have

$$Acceleration \ of \ P = \frac{du}{dt} + \frac{u^2}{r} + h \alpha +$$

$$h w^2 + 2 u w \sin \theta$$

The plus signs indicating geometrical addition.

The components are shown in figure 23.
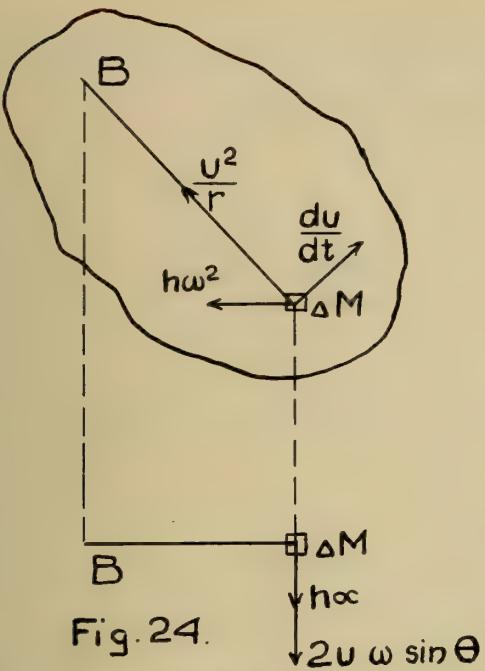
Fig. 24.

Let ΔM denote the particle of mass at P; then the accelerating forces at P are found by multiplying the acceleration components by ΔM. By D'Alembert's principle these forces reversed hold the external forces in equilibrium. We have now to sum each of these for the entire mass. The forces $\Delta M h \propto$ $\Delta M \ 2 u \omega \sin \theta$ and $\Delta M \frac{u^2}{r}$ act either perpendicular to the plane of the governor's motion, or through the center of rotation B. Hence they do no work in moving the governor and may be dropped from further consideration. As the tangential acceleration $\frac{du}{dt}$ is very small the effect of the force $M \frac{du}{dt}$ is negliable, compared with the large centrifugal force which will no be considered.

By summation we get
$$\Sigma \, dF_r = \Sigma \Delta M h \, \omega^2 = \omega^2 \Sigma \Delta M h$$

But $\quad \Sigma \Delta M h = M h$

Therefore $\quad F_r = M h \, \omega^2$

Although the forces $2 u \omega \sin \theta \Delta M$ and $h \propto \Delta M$ act in a plane perpendicular to the governor's motion and do not effect the governoring action yet they exert an influence upon the motion of the engine. These inertia forces must be overcome by the turning moment of the crank shaft, just as the inertia of the fly-wheel and other moving masses must be overcome.

2. <u>Equations for Reducing Forces.</u>

A general equation which can be applied for reducing forces will now be derived. The principal of force reduction has been given as the equality of work. Original forces will be denoted by Roman capital and the reduced forces by corresponding script capitals. The velocity of the point of application of the original forces will be denoted by 'v' and the velocity of the point to which they are reduced by v.
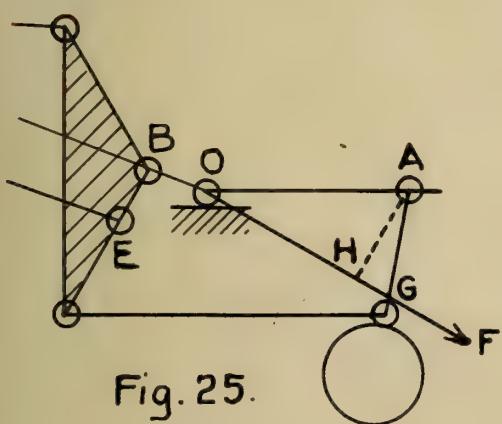
Take for illustration the centrifugal shaft governor shown schematically in figure 25. Required to reduce force F to point E, perpendicular to BE. From A drop a perpendicular AH, to action line OG. Take H as the point of application of

Fig. 25.

force F. To find the work of force
F for an instant of time, its mag-
nitude is multiplied by the velocity
of H and the product by a diff-
erential element of time.

$$Work = Force \times ds$$

But $ds = v'dt$

Therefore $W = Fv'dt$

This equation gives the work of the
force F for an interval of time dt.
The work of the reduced force at E
is $Fvdt$,

Therefore $Fv'dt = Fvdt$

This equation can be used for re-
ducing all of the forces which a-
rise when a mechanism is in motion.
The magnitude and method of determin-
ing these forces has already been
given.

Reduction of the Link Weight considered as a Force. Using our previous equation we have at once

$$G_n v dt = G_n v' dt$$

$$G_n = \frac{G_n v'}{v}$$

The sign of this force is positive or negative if it helps or hinders the existing motion, Or, in other words, it is positive when it has a component acting in the direction of the path of the center of gravity.

Reduction of the Radial Inertia Force. Let F_r = radial inertia force, then

$$F_r v dt = F_r v' dt$$

$$F_r = \frac{F_r v'}{v}$$

The sign is determined in the same way as in the preceding case.

Reduction of Tangential Inertia Force.

Let F_t = tangential inertia force, then

$$F_t \, v \, dt = F_t \, v' \, dt$$

$$F_t = \frac{F_t \, v'}{v}$$

Reduction of Angular Inertia Resultant

Let $I\alpha$ = moment of inertia couple.

$\quad\quad \omega$ = angular velocity of link
$\quad\quad\quad\quad$ under consideration

Then

$$F \, v \, dt = I\alpha \, \omega \, dt$$

$$F = \frac{I\alpha \, \omega}{v}$$

The sign of this force is found from the consideration that each link has an angular acceleration about its own center of gravity. If the inertia couple thus developed helps the existing motion the value of the reduced force is positive, if not the reverse is true.

IV – <u>Applications</u> <u>to</u> <u>Existing</u> <u>Mechanism</u>.

1. <u>Method</u> <u>of</u> <u>determining</u> <u>data</u>, <u>weights</u>
etc, <u>tabulated</u>.

The mechanism selected for the
application of this theory is the
fly-ball governor on the York Ice
Machine in the Mechanical Engineer-
ing Laboratory. The analysis of this
problem has already been given,
it remained only to determine the
necessary data for a complete sol-
ution

The first step taken was to
make accurate kinematic drawing
of the governor gear in its lowest
position. All of the links were then
carefully weighed and their centers
of gravity located. The moments of
inertia of all links having a ro-
tation either about a fixed or an
instantaneous axis were then de-
termined by the pendulum methods
already described. Data taken from

link No. 1, the fly weight, will serve as an example

Link No. 1

Weight = 11.9 pounds with pin.
Diameter of eye = 5/16 inch.
Center of gravity = 12 inches from center of eye.

1st Observation — 50 oscillations in 1 min.
2nd   — 101   2 .
3rd   — 101   2 .

The following calculations were then made.

$$Mass = \frac{11.9}{32.2} = .37 \text{ gee pounds.}$$

$$T = \frac{120}{101} = 1.19 \text{ seconds.}$$

$$h = 12\,5/16 \text{ inches} = 1.003 \text{ feet.}$$

$$K_s^2 = \frac{T^2 h g}{\pi^2} = \frac{\overline{1.19}^2 \times 1.003 \times 32.2}{4\pi^2} = 1.15$$

$$\overline{K}^2 = K_s^2 - d^2 = 1.15 - \overline{1.003}^2 = .15$$

$$K'^2 = \overline{.15}^2 + \overline{1.0015}^2 = 1.13$$

TABLE I

Link No	T sec.	h	Mass	$K_s'^2$	K'^2	$\bar{K}^2$	I'	$\bar{I}$
1	1.19	1.003	.37	1.15	1.13	.15	.425	.0555
2	.904	.429	.054	.287	.286	.1	.155	.0054
3-4			2.94					
5	.94	.0678	.0777	.0488	.0443	.0438	.00344	
5_1	.6	.151	.0435	.0443	.0323	.0213	.00143	
5_2	calculated		.087					
5_3	1	.628	.386	.514	.469	.129	.181	
$5_3'$	1.27	1.056	.386	1.38	1.28	.26	.495	
5 Total							.186	
5 Total							.5	
6			.087					
7			.053					
8	.656	.151	.123	.052	.039	.029	.0048	

$$\overline{I} = .37 \times .15 = .0555$$

$$I' = .37 \times 1.13 = .425$$

Similar calculations were made for each link and the results tabulated in table I . In this table K'_3 is the radius of gyration with respect to the axis of suspension. The determination for link No. 5 was made by separating it into three pieces. This link has an adjustable weight and the moment of inertia was found for two positions of this weight.

2. Method of Reducing Masses.

We can now reduce all the masses of the governor that accumulate kinetic energy to a single mass concentrated at the center of the ball. For example the reduced mass of link No. 1 $= \dfrac{I'}{r^2} = \dfrac{.425}{1.17^2} = .31$

Table II contains the calculations made for the first position

TABLE II

Link No	Mass	I	I′	Formula for Reduced Mass	Reduced Mass
1			.425	$2 \times .425 \times \dfrac{1}{1.17^2}$	.62
2	.054	.0155	.0505	$2 \times .0505 \times \left[\dfrac{75}{.935 \times 1.17} \right]^2$	.0475
3&4	2.94			$2.94 \times \left[\dfrac{.75 \times .875}{.935 \times 1.17} \right]^2$	1.058
5_1			.181	$1.81 \times \left[\dfrac{.75 \times .875}{.935 \times 1.17 \times .375} \right]^2$	.467
5_2			.495	$.495 \times \left[\dfrac{.75 \times .875}{.935 \times 1.17 \times .375} \right]^2$	1.27
6	.087				
7	.053			$.14 \times \left[\dfrac{.75 \times .875 \times 48}{.935 \times 1.17 \times .375} \right]^2$	.0825
8			.0048		
9			.0048	$2 \times .0048 \left[\dfrac{.75 \times .875 \times 48}{.935 \times 1.17 \times .375 \times 417} \right]^2$	.0326
				Total	2.307
				Total	3.1106

The total range of the governor was divided into five parts and similar calculations were made for each position. The final results are tabulated in table

TABLE III

Position No.	A	B
1	2.307	3.1106
2	2.596	3.541
3	2.926	3.934
4	3.177	4.398
5	3.509	4.89

The values in the column headed A are those obtained by using the smallest value of the moment of inertia of link No. 5. Those in column B by using the larger value.

A graphical representation of the results was made by dropping ordinates from the center of the ball

and laying off on these lines seg-
ments proportional to the reduced
mass at that point. A smooth curve
through these points is called the
curve of reduced masses. This curve
together with the necessary graphical
construction is shown on Plate I
on the following page. Curve MN
represents values in column A of
table   and M'N' in column B of
the same table. The small circles
show the exact positions of the points
as found by calculation.

3. <u>Equation</u> of <u>Motion</u> of <u>Fly-ball</u>
   <u>Governor</u>.

In order to determine the
inertia forces arising in a fly-
ball governor, it is necessary to
know the total acceleration of the
flyweight. This requires that the
angular acceleration and the vel-
ocity of occilation must be known
for any governor position, and

PLATE I
CURVES OF
REDUCED MASS
OF YORK ICE MACH. GOV.

Reduced Mass in Gee Pounds
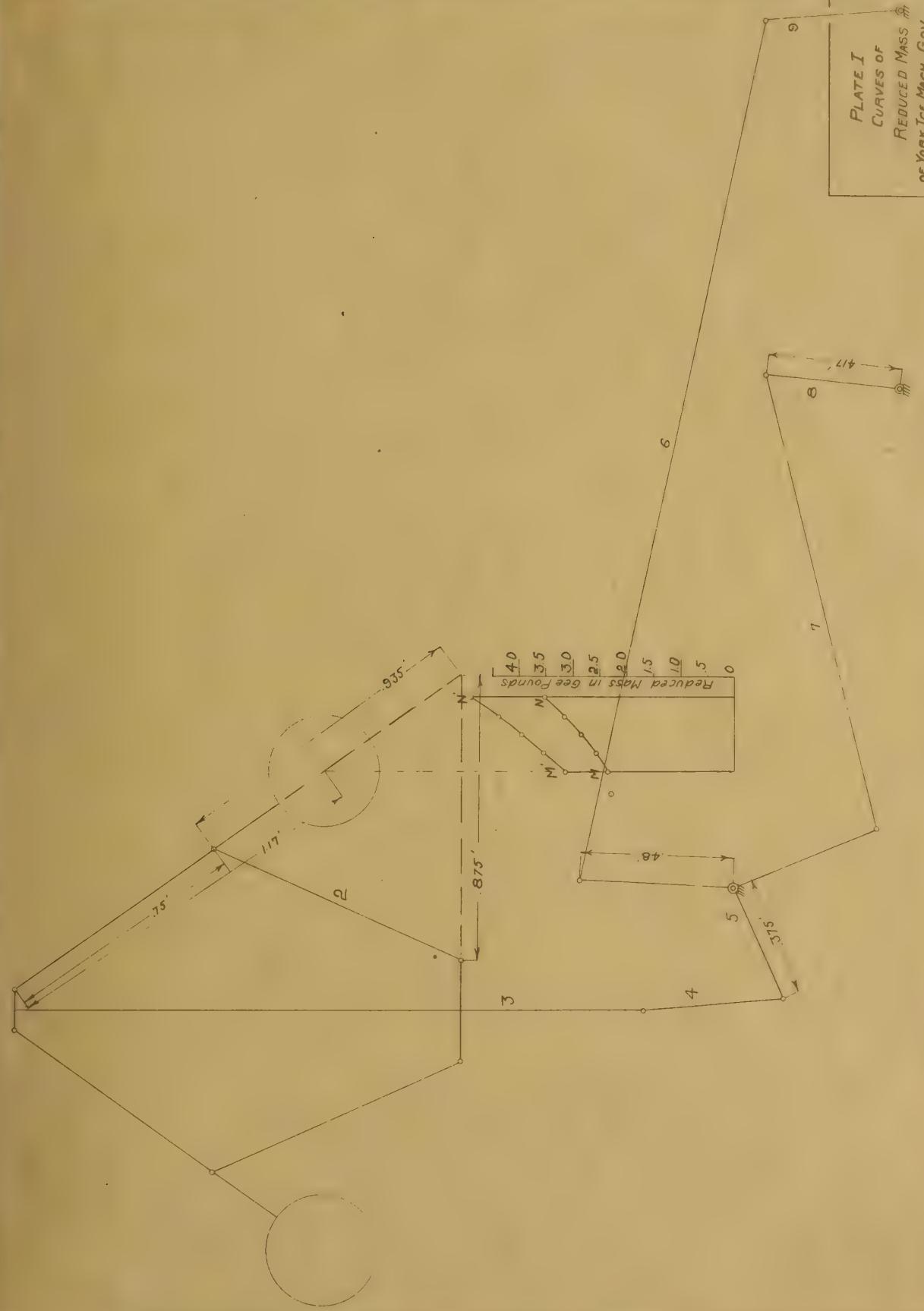
for any change of load on the engine.
The equation of motion of the fly-
ball governor is required for this
purpose. The following discussion
and the derivation of the equation
of motion is due to Tolle.

Before deriving the equation
of motion, several other considera-
tions which have a direct bearing
upon its development will be taken
up. The most important of these
is the C curve.
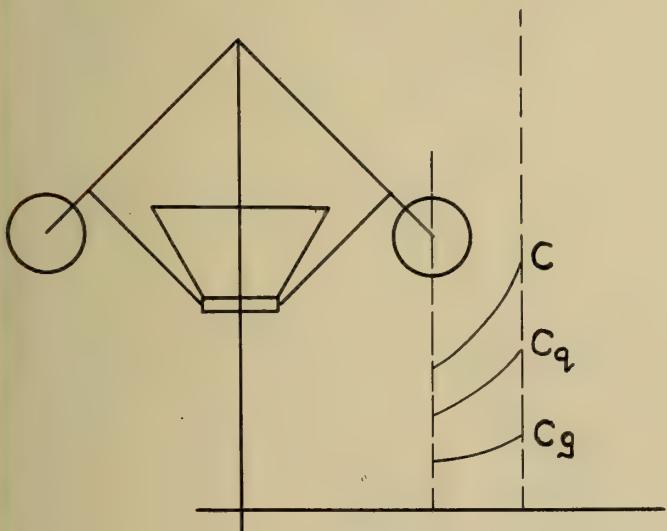
From figure 26
it is evident
that a certain
force is nec-
essary to hold
the governor
in a given
position; this
force may be
considered to
act through

Fig. 26.

the center of gravity of the fly-weight,

and varies with the position chosen.
Let ordinates be dropped from the
center of gravity of the pendulum
to a horizontal line, and distances
be laid off vertically to some scale
equal to the required force. The
points thus found may be con-
nected by a smooth curve, which
will be called the C curve. The
force thus required may be con-
sidered as being made up of two
parts, the first C_g, which holds the
pendulum in equilibrium and the
second C_q which holds the weight.
If C represents the total force then

$$C = C_g + C_q$$

The C_g and the C_q curves may
be drawn for the entire governor
range and the sum of the ordinates
at any point will determine a point
on the C curve.

  The problem of constructing

the C curve now reduces itself to
one of graphical statics.
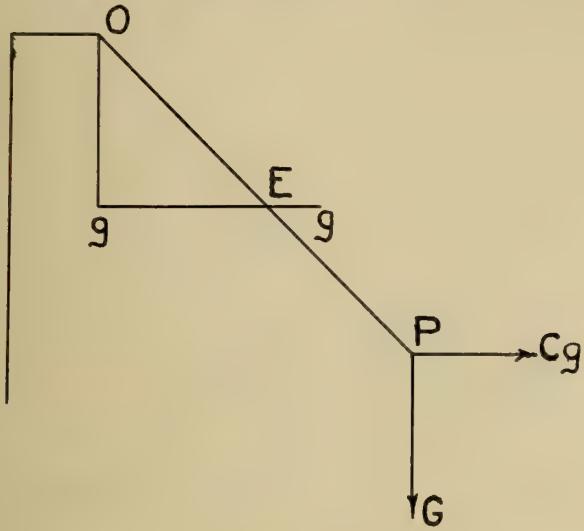
Fig. 27.

If P is the cen-
ter of gravity
of the pendulum
and G its
weight then the
moment of C_g
about O must
be equal to the
moment of
about O. From
O draw the

vertical line Og and make it equal
to G, through g draw gg horizontally
until it intersects OP at E, gE, then,
is the required force for the chosen
position of the pendulum.

The force C_g must hold the
weight Q in equilibrium. Resolve Q
into two components, one along the
line BH and the other horizontal. The
forces S and C_g must now be in
equilibrium with the pin reaction
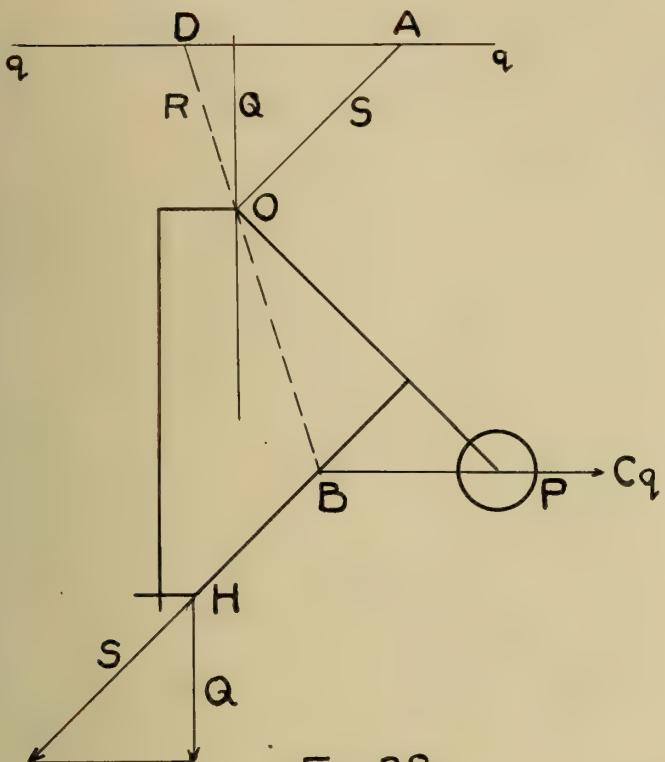
at O. If B
is the point
where C_q
intersects
the action
line of S
then BO
is the action
line of the
pin reaction
R. Through
O draw OA
parallel to

Fig. 28.

BH and lay off OA equal to S. Through
A draw qq horizontal and produce
BO until it intersects qq at D. Then
AD gives the value of C_q. This con-
struction lends itself readily to the
complete determination of the C_q
and the C_q curves. For each posi-
tion only three new lines need be
drawn.

In this discussion friction
has not been taken into account.

But for any governor where the size
of the pins are known the construc-
tion may be modified by taking
the pin reactions tangent to the
proper friction circles. It is also
important to note that the position
of the governor axis has not been
taken into account in this con-
struction; in other words, the posi-
tion of the governor axis has no
effect on the form of the C
curves.

When the governor is in a
position of equilibrium, it is held
there by centrifugal force due to
the rotation of the pendulum about
the governor axis. From the fore-
going this force C must be equal
to $C_g + C_q$.

Let $x =$ distance of center of gravity
        from axis.
$M =$ mass of pendulum.
$n =$ revolutions per minute.
$\omega =$ angular velocity.

Then $\omega = \dfrac{2\pi n}{60} = \dfrac{\pi n}{30}$

And $C = Mx\omega^2$ or $\omega^2 = \dfrac{C}{Mx}$.

Whence $\omega = \sqrt{\dfrac{C}{x} \cdot \dfrac{1}{M}}$

Therefore $n = \dfrac{30\,\omega}{\pi} = \dfrac{30}{\pi}\sqrt{\dfrac{C}{x} \cdot \dfrac{1}{M}}$

From this equation the revolutions per minute may be calculated when the centrifugal force C and the distance x are known. By drawing separate curves for C_g and C_q the effect of each upon the centrifugal force may be determined.
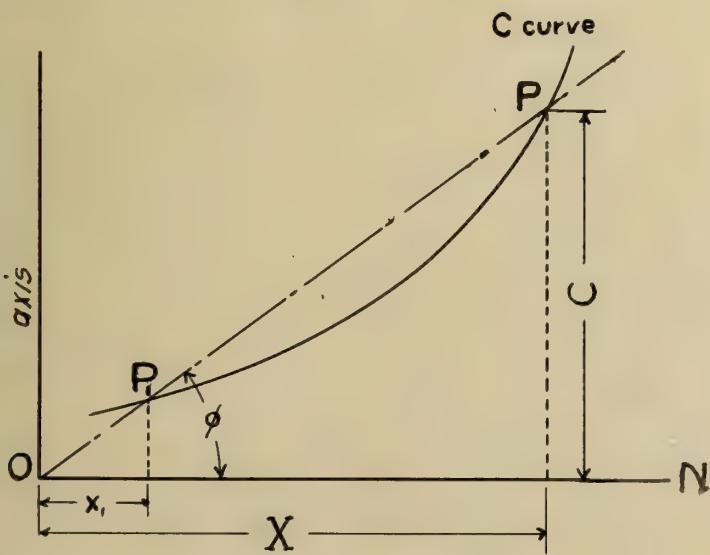

Fig. 29.

In figure 29 a ray is drawn from O to a point P on the C curve. Denoting by ϕ the angle PON

we have; $\dfrac{c}{x} = \tan \varphi$

Substituting in the equation

$$\omega = \sqrt{\dfrac{c}{x} \cdot \dfrac{I}{M}} \quad \text{we get}$$

$$\omega = \sqrt{\dfrac{I}{M}} \cdot \sqrt{\tan \varphi}$$

And $\quad n = \dfrac{30}{\pi} \sqrt{\dfrac{I}{M}} \cdot \sqrt{\tan \varphi}$

The form of the C curve illustrated many of the governor's characteristics. If it intersects the ray OP in another point as P_1 then

$$\dfrac{c_1}{x_1} = \dfrac{c}{x}$$

and therefore from the preceding equation we have that for both positions of the governor the angular velocity is the same. If the C curve is a straight line through O then the governor has the same angular velocity for all

positions because $\frac{c}{x}$ is constant.
Such a governor is called astatic
and cannot be used for the
regulation of steam engines. It
will change its configuration
from one extreme position to
another with the slightest change
in angular velocity. A governor
can only be used when it has
a different speed for each position
it is then said to be static. If
the speed changes the governor will
seek its new position and in
passing from one to the other
it performs its useful function.

Another characteristic of the
C curve essential to a good gov-
ernor should be noted. As the
distance x is increased the curve
should have such a form that
the angle ϕ constantly increases.
If this is the case the angular vel-
ocity will increase with a rising
pendulum. If, on the other hand,

the angle φ decreases with increase
of the abscissae x then the gov-
ernor will have the smallest ang-
ular velocity for its highest posi-
tion. It is evident, therefore, that
the C curve shows clearly how
the angular velocity varies with
the governor position. The C curve
also distinguishes the character
of the governor, if it is astatic
or static, useful or not useful.
The investigation or design of a
governor should therefore begin
with the construction of the C
curves.

The function of a governor
is to keep the speed of the motor
constant with any load. This re-
quirement cannot be ridigly ful-
filled by a fly-ball governor,
because as already shown it
must have a certain degree of
stability, that is, each position
corresponds to a different an-

gular velocity. The increase of an-
gular velocity from the lowest
to the highest governor position
should therefore be as small as
possible. This change in angular
velocity is denoted by $\delta\omega$ and δ
is called the coefficient of fluctua-
tion. If ω_1 and ω_2 denote the
smallest and the largest angular
velocities, then $\delta = \dfrac{\omega_2 - \omega_1}{\omega m}$ where
ωm is the arithmetical mean of
ω_1 and ω_2, that is, $\omega_m = \dfrac{\omega_2 + \omega_1}{2}$.
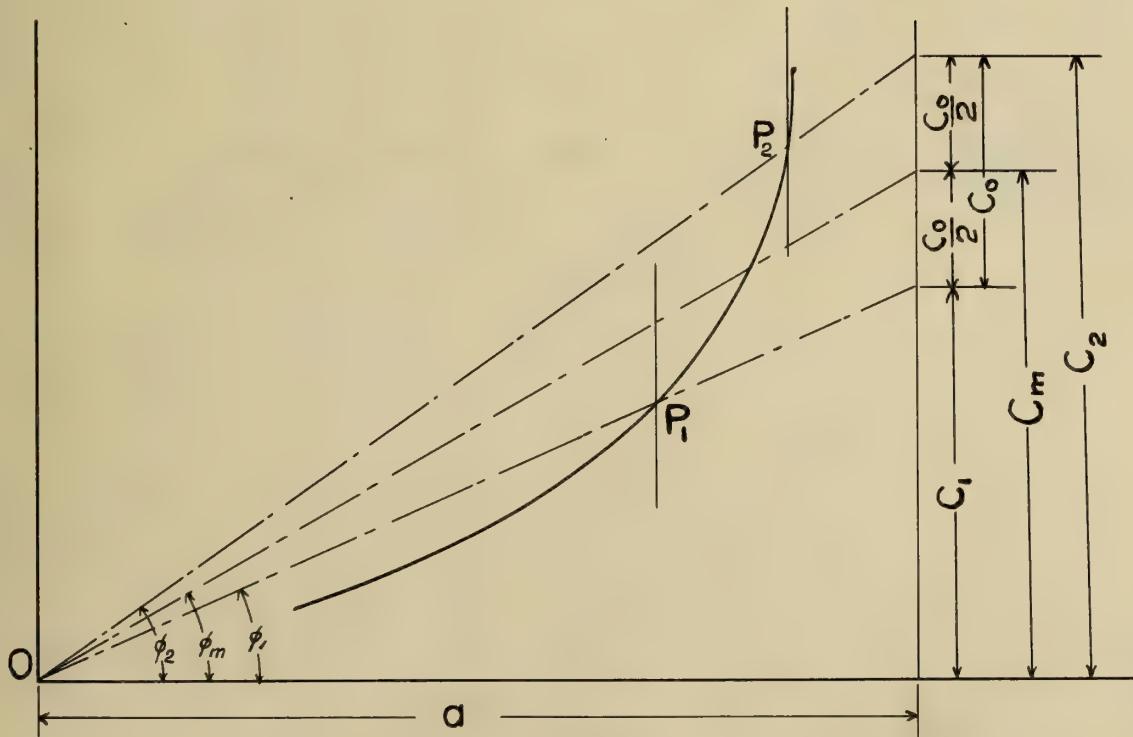
Fig. 30.

$$\delta = \frac{\omega_2 - \omega_1}{\omega_m} = \frac{(\omega_2 - \omega_1)\left(\frac{\omega_2 + \omega_1}{2}\right)}{\omega^2_m} = \frac{\omega_2^2 - \omega_1^2}{2\omega_m^2}$$

But $\omega_2^2 = \frac{1}{M}\tan \varphi_2 \quad \omega_1^2 = \frac{1}{M}\tan \varphi_1$

$$\omega_m^2 = \frac{1}{M}\tan \varphi_m$$

Therefore $\delta = \frac{\tan \varphi_2 - \tan \varphi_1}{2 \tan \varphi_m}$

In figure 30 take a equal to the distance of the center of gravity of the pendulum from the axis in its highest position. Through P_1 and P_2 draw rays from O until they intersect a perpendicular through a cutting from it the segments c_1 and c_2

Take $C_m = \frac{C_1 + C_2}{2}$

Then $C_0 = C_2 - C_1$

$\tan \varphi_1 = \frac{c_1}{a}$, $\tan \varphi_2 = \frac{c_2}{a}$, $\tan \varphi_m = \frac{C_m}{a}$

Therefore $\delta = \dfrac{c_2 - c_1}{2 c_m} = \dfrac{c_0}{2 c_m}$

$\qquad$ or $\quad c_0 = 2 \delta c_m$.

If the coefficient of regulation
and the extreme governor position
is given, c_0 may be calculated
and laid off on a perpendicular.
If a ray is then drawn through
this point it will intersect the c
curve in the lowest position. The
range of the governor is thus de-
termined.

$\qquad$ We will now consider the
action of the governor when the
load on the motor is changed.
As already stated, for each gov-
ernor position there is a corres-
ponding load on the motor. The
lowest position gives the greatest
turning moment and the highest
the smallest – very nearly zero.
This being the case we may plot
a load curve with the position
of the center of gravity of the

pendulum as abscissae and load as
ordinates. (The ordinates may be
expressed as horse power, turning
effort or mean effective pressure.)
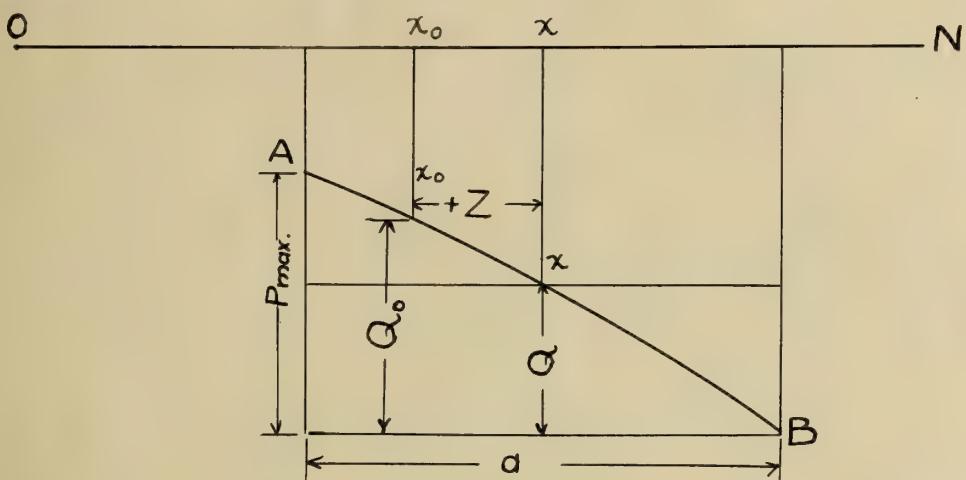
Fig. 31.

Figure 31 shows the resulting load
curve AB. This curve will bend
slightly at the top for two reasons.
First because for the lowest positions
a displacement of the pendulum
gives smaller displacements along
the governor axis than for the ex-
treme higher positions. Secondly,
because at large loads a change
in the cut-off has less effect on

the mean effective pressure than at
light loads. However in order to
simplify the discussion the load
curve will be assumed to be a
straight line.

If the load on the motor is
Q_0 the corresponding position of the
center of gravity of the governor
pendulum is in the axis $x_0 x_0$.
Consider now that the load should
be suddenly reduced to Q. A dis-
turbance of equilibrium will ensue.
As the governor requires some
perceptible time to get into its
new position xx, there will be
an excess turning moment on
the engine shaft $Q_0 - Q$. This excess
is used in accelerating the shaft and
hence also the governor. With an in-
crease in angular velocity there will
be an increase in centrifugal force
which will shift the governor into
its new position. If T is the time
in seconds required by the motor

to acquire its normal speed with no load and greatest cut-off, ω the angular velocity and α the angular acceleration of the governor we have:

$$\alpha_{max} = \frac{\omega}{T}$$

This is the maximum angular acceleration of the governor when the load on the motor is suddenly changed from full load to no load.

If the load curve is a straight line then the excess of turning effort over the resisting moment is proportional to z, where z is the distance of the pendulum from the equilibrium position — see figure . Consequently the acceleration of the governor is proportional to z. If a is the range of the governor measured horizontally we have

$$\alpha = \frac{\alpha_{max}}{a} z = \frac{\omega}{Ta} \cdot z$$

the acceleration for any change in load. The abscissa z is positive when measured toward the governor axis and negative when measured in an opposite direction.

The equation of motion of the fly-ball governor will now be deduced. Let the governor move from its position $x_0 x_0$ to its new position of equilibrium $x x_1$, see figure 32.
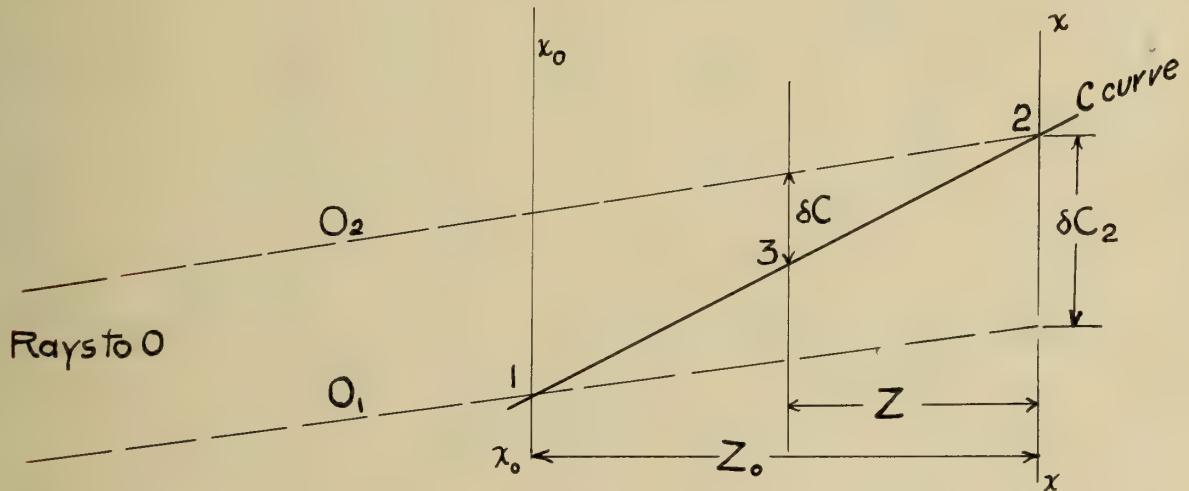

Fig. 32.

We will make the following as-sumptions;

1. Load curve is a straight line.

2. C curve is a straight line.

3. That the governor range is so small as compared with its distance from the governor axis x that the rays $o1$ and $o2$ may be considered parallel.

Also let λ = change in load

Therefore $\lambda = \dfrac{Z_0}{a}$

If, as soon as equilibrium were disturbed, the angular velocity would rise instantaneously to that corresponding to ray $o2$ and the center of gravity of the pendulum had only moved to a distance from position 3. A new equilibrium position force δC would be available for accelerating the governor masses. For at the position indicated, the force C measured to the C curve will hold the governor in equilibrium. The excess force δC is proportional to Z. Giving to C_0 its previous value we have

$$\frac{\delta C}{C_0} = \frac{Z}{a} \qquad \text{or} \qquad \delta C = \frac{C_0}{a} \cdot Z$$

As in reality the rays are not parallel this equation is only approximately true.

$$\text{But } C_0 = 2 C_m \delta$$

$$\text{Therefore } \delta C = \frac{2 \delta \cdot C_m}{a} \cdot z$$

δC is, however not the only force present to accelerate the governor masses. Actually the new angular velocity is not acquired instantaneously. If it takes t seconds for the governor to arrive at position 3 there has been an increase in angular velocity of

$$\Delta \omega = \int \alpha \, dt$$

Hence there is also an increase of centrifugal force δC_1

$$\delta C_1 = (\omega + \Delta \omega)^2 Mx - \omega^2 Mx = 2\omega \cdot \Delta \omega \, Mx$$

where M is the mass of pendulum

Now let $C_m = M \times \omega^2$

Then $\delta C_1 = 2 \dfrac{C_m}{\omega} \cdot \Delta \omega$

$$= 2 \dfrac{C_m}{\omega} \int \alpha \, dt.$$

The new equilibrium position of the governor requires an increase of centrifugal force δC_2

$$\delta C_2 = \dfrac{C_0}{a} Z_0 = C_0 \lambda = \lambda 2 \delta C_m$$

We have, therefore, the excess of centrifugal force which is available for accelerating the governor masses

$$\delta C_1 - \delta C_2 = \dfrac{2 C_m}{\omega} \int \alpha \, dt - \lambda 2 \delta C_m$$

The total force which is available for acceleration is

$$P = \delta C + \delta C_1 - \delta C_2 \quad \text{or}$$

$$P = \dfrac{2 \delta C_m}{a} \cdot Z + \dfrac{2 C_m}{\omega} \int \alpha \, dt - \lambda 2 \delta C_m$$

But $\quad \alpha = \dfrac{\omega}{Ta} \cdot z$

Therefore

$$P = \dfrac{2\delta C_m}{a} \cdot z + \dfrac{2C_m}{Ta} \int z \, dt - \lambda z \delta C_m$$

Consider all of the masses of the governor gear replaced by a single mass which is moved horizontally by the force P. Let U_r equal the reduced mass and b its acceleration. Then

$$b = \dfrac{P}{U_r} = -\dfrac{d^2 z}{dt^2} = \dfrac{1}{U_r}\left[\dfrac{2\delta C_m}{a} z + \dfrac{2C_m}{Ta} \int z \, dt - \right.$$

$$z \lambda \delta C_m$$

$\dfrac{d^2 z}{dt^2}$ is negative because the acceleration decreases proportional as z increases. Differentiating the preceding equation with respect to t we have

$$\dfrac{d^3 z}{dt^3} + \dfrac{2\delta C_m}{a U_r} \dfrac{dz}{dt} + \dfrac{2C_m}{Ta U_r} \cdot z = 0$$

which is the differential equation of motion of the fly-ball governor. Let $\dfrac{2C_m}{a U_r} = A$ a constant we have

$$\frac{d^3z}{dt^3} + \delta A \frac{dz}{dt} + \frac{A}{T} \cdot z = 0 \quad \text{- - - - - - - } 1.$$

This equation shows that the velocity and acceleration of the governor pendulum depends upon

1st on $A = \dfrac{2Cm}{aUr}$

2nd on δ the coefficient of regulation

3rd on T

The quantity T depends upon the mass of the engine such as the fly wheel, counter weight and other masses which have to be accelerated.

The solution of the differential equation 1 is

$$Z = C_1 \, \ell^{w_1 T} + C_2 \, \ell^{w_2 T} + C_3 \, \ell^{w_3 T} \text{ - - - - - - } 2.$$

Where w_1, w_2 and w_3 are the roots of the cubic equation:

$$w^3 + \delta A w + \frac{A}{T} = 0 \quad \text{- - - - - - - - - - - - } 3$$

and C_1, C_2 and C_3 are constants de-

termined by the following conditions

When $t = 0$ $Z = Z_0$

$t = 0$ $\dfrac{dz}{dt} = 0$

$t = 0$ $\dfrac{d^2z}{dt^2} = 0$

From Descartes rule of signs eq-
uation 3 has one real root $\omega_1 = -2p$
and two imaginary ones $\omega_2 = p + qi$
and $\omega_3 = p - qi$. Substituting these
latter values in equation 2 we obtain

$$Z = C_1 e^{\omega_1 t} + e^{pt}(C_2 \cos qt + C_3 \sin qt) \text{ - - - - } 4.$$

The roots of equation 3 are

$$\omega_1 = \sqrt[3]{-\dfrac{A}{2T} + \sqrt{\left(\dfrac{A}{2T}\right)^2 + \left(\dfrac{\delta A}{3}\right)^3}}$$

$$+ \sqrt[3]{-\dfrac{A}{2T} - \sqrt{\left(\dfrac{A}{2T}\right)^2 + \left(\dfrac{\delta A}{3}\right)^3}}$$

$$q = \left[\sqrt[3]{-\frac{A}{2T} + \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}} \right.$$

$$\left. -\sqrt[3]{-\frac{A}{2T} - \sqrt{\left(\frac{A}{2T}\right)^2 + \left(\frac{\delta A}{3}\right)^3}} \right] \sqrt{3}$$

Applying the three conditions given for determining the constants c_1, c_2 and c_3 to equation 4 and solving we have:

$$C_1 = \frac{p^2 + q^2}{9p^2 + q^2} \cdot Z_0$$

$$C_2 = \frac{8p^2}{9p^2 + q^2} \cdot Z_0$$

$$C_3 = \frac{2q^2 - 6p^2}{9p^2 + q^2} \cdot \frac{p}{q} \cdot Z_0$$

Substituting $-2p$ for ω, in equation 4 we have

$$Z = C_1 e^{-2pt} + e^{pt} (C_2 \cos qt + C_3 \sin qt)$$

Differentiating, we have, for the velocity.

$$\frac{dZ}{dt} = C_4 e^{-2pt} + e^{pt} (C_5 \cos qt + C_6 \sin qt)$$

where $C_4 = -2pC_1$, $C_5 = pC_2 + qC_3$

$$C_6 = pC_3 - qC_2$$

The acceleration is

$$\frac{d^2z}{dt^2} = C_7 e^{-2pt} - e^{pt}(C_8 \cos qt + C_9 \sin qt)$$

where $C_7 = C_8 = 4p^2C_1$, $C_9 = 2pqC_2 + C_3(q^2 - p^2)$

The C curve for the governor was determined experimentally. Simultaneous observations were made of the speed of the governor and the position of the muff. Beginning with the lowest position the speed was increased enough to allow the governor to pass through its entire range. The speed was kept constant in any one position by means of the throttle. Observed values are given in following table, the position of the stem being measured from a fixed point.

Time for 100 R.P.M.	Position of stem
51 sec.	$\frac{15}{16}''$
$50\frac{1}{5}$ "	$1\frac{7}{16}''$
49 "	$1\frac{5}{8}''$
48 "	$2''$

Now $C = M \times \omega^2 = M \times \left(\frac{2\pi N}{60}\right)^2 = \frac{M \times \pi^2 N^2}{900}$

The distance of the center of gravity of the fly ball from the stem denoted by x was measured from the drawing for each position. All other quantities being known, C can be found.

For position 1, $C = 85$ pounds.

2, $C = 92$

3, $C = 98.6$

4, $C = 106$

The C curve shown on Plate II was plotted with these values. From this curve $C_1 = 85$ pounds $C_2 = 118.5$ pounds.

Therefore $C_m = \frac{85 + 118.5}{2} = 100$ approximately

$$C_0 = C_2 - C_1 = 118.5 - 85 = 33.5 \text{ pounds}$$

$$\delta = \frac{C_0}{2C_m} = \frac{33.5}{2 \times 100} = 16.7\%$$

From the curve of reduced masses, M for the position of greatest cut-off is 3.78 gee pounds. M is the mass reduced to the center of the ball 14 inches from the pin. Reducing M to the center of gravity of the pendulum, 12 inches from the pin we have

$$M_r = \frac{M \overline{14}^2}{\overline{12}^2} = \frac{3.78 \times 196}{144} = 5.15 \text{ gee pounds}$$

But M_r is not identical with U_r, because U_r has a velocity $\frac{dz}{dt}$ and M_r a velocity $b\frac{d\theta}{dt}$.
Therefore we have;

$$\frac{1}{2} U_r \left(\frac{dz}{dt}\right)^2 = \frac{1}{2} M_r \left(b \frac{d\theta}{dt}\right)^2$$

But $\frac{dz}{dt} = b \frac{d\theta}{dt} \cos \alpha$

Therefore $U_r = \dfrac{M_r}{\cos^2 \alpha} = \dfrac{5.15}{\cos^2 \alpha} = 9$

U_r may be found graphically when M is known. In figure 33 lay off SC equal to M_r, draw CA perpendicular to SC until it intersects the horizontal AS. From A erect a vertical to B. Then SB
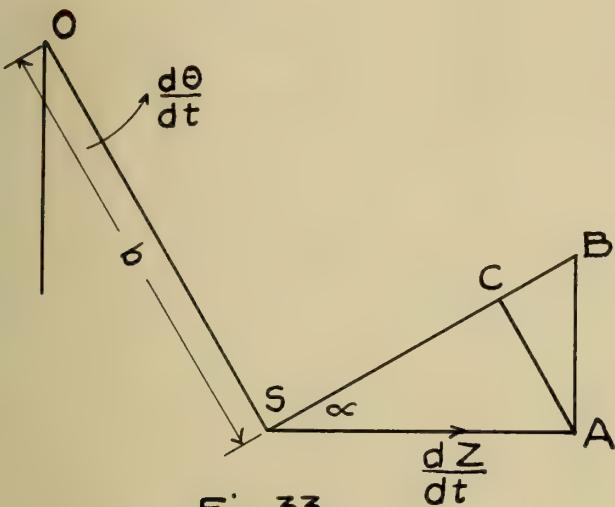
Fig. 33.

represents U_r to the same scale that SC represents M_r.

The horizontal range of the fly-ball was .115 feet and the time T as observed was two seconds. All quantities involved in the equation are known except Z_0 which depends upon the change of load. Assuming that the engine is running at full load and half of the load is suddenly thrown off, we will

consider the following problem. To determine all forces existing in the governor gear at the end of .2 of a second after load has been removed and to reduce these forces to the center of gravity of the fly-ball with a horizontal line of action.

$$A = \frac{2\,C_m}{a\,U_r} = \frac{2 \times 100}{.115 \times 9} = 256$$

$$\delta A = 256 \times .167 = 42.8$$

$$\omega_1 = \sqrt[3]{\frac{256}{2 \times 2} + \sqrt{\left(\frac{256}{4}\right)^2 + \left(\frac{42.8}{3}\right)^3}} + \sqrt[3]{-\frac{256}{4} + \sqrt{\left(\frac{256}{4}\right)^2 + \left(\frac{42.8}{3}\right)^3}}$$

$$\omega_1 = -2.59 \qquad p = 1.29$$

$$q = 8.01 \times \sqrt{3} = 13.9$$

$$C_1 = \frac{\overline{1.29}^2 + \overline{13.9}^2}{+ \overline{13.9}^2}, \quad Z_0 = \frac{2.22 \times 194}{20 \times 194} \times .0575 = .0528$$

$$C_2 = \frac{8 \times 2.22}{214} \times .0575 = .00477$$

$$C_3 = \frac{388 - 13.3}{214} \times .0575 \times \frac{1.29}{13.9} = .00108$$

Let $t = .2$

$$Z = \frac{.0528}{e^{.596}} + e^{.298}\left(.00477 \cos 13.9 \times .2 + .00108 \sin 13.9 \times .2\right)$$

$$= \frac{.0528}{1.81} + 1.35\left(.00477 \times -.933 + .00108 \times .358\right)$$

$$= .0292 + 1.35\left(-.00445 + .00039\right)$$

$$= .0292 - .00548 = .0237 \text{ feet.}$$

$$C_4 = -2 \times 1.29 \times .0528 = -.1575$$

$$C_5 = 1.29 \times .00477 + 13.9 \times .00108 = .0221$$

$$C_6 = 1.29 \times .00108 - 13.9 \times .00477 = -.06479$$

$$\frac{dz}{dt} = -\frac{.1575}{1.81} + 1.35\left(-.0221 \times .933 - .06479 \times .358\right)$$

$$= -.087 - .059 = -.146 \text{ feet per second}$$

$$C_7 = C_8 = 4 \times 2.22 \times .0528 = .469$$

$$C_9 = 2 \times 1.49 \times 13.9 \times .00477 + .00108\left(194 - 2.22\right)$$

$$= .198 + .208 = .406$$

$$\frac{d^2 Z}{d t^2} = \frac{.469}{1.81} - 1.35(.469 \times -.933 + .406 \times .358)$$

$$= .259 - .394 = -.135 \text{ feet per second}$$
per second.

For the change of load let us assume the equilibrium position to be $\frac{a}{2}$ or $\frac{.115}{2} = .0575$ feet from the point M, Plate II,. At the end of .2 of a second $Z = .0237$ feet or the center of gravity of the fly-ball is $.0575 - .0237 = .0358$ feet from M. Lay off $MN = .0358$, at N erect a perpendicular until it intersects the path of the center of gravity G of the pendulum. The configuration of the governor is now fixed.

The scale of distance is 1 inch equals .166 feet, let scale of velocity be one inch $= .09$ feet per second. Then the scale of acceleration equals $\frac{.09^2}{.166} = .0486$ feet per second per second. The horizontal component of G's velocity being known we can construct the velocity polygon

PLATE II
CONSTRUCTIONS
FOR REDUCING FORCES
OF YORK ICE MACH GOV

Force in pounds
90 75 60 45 30 15

Curve
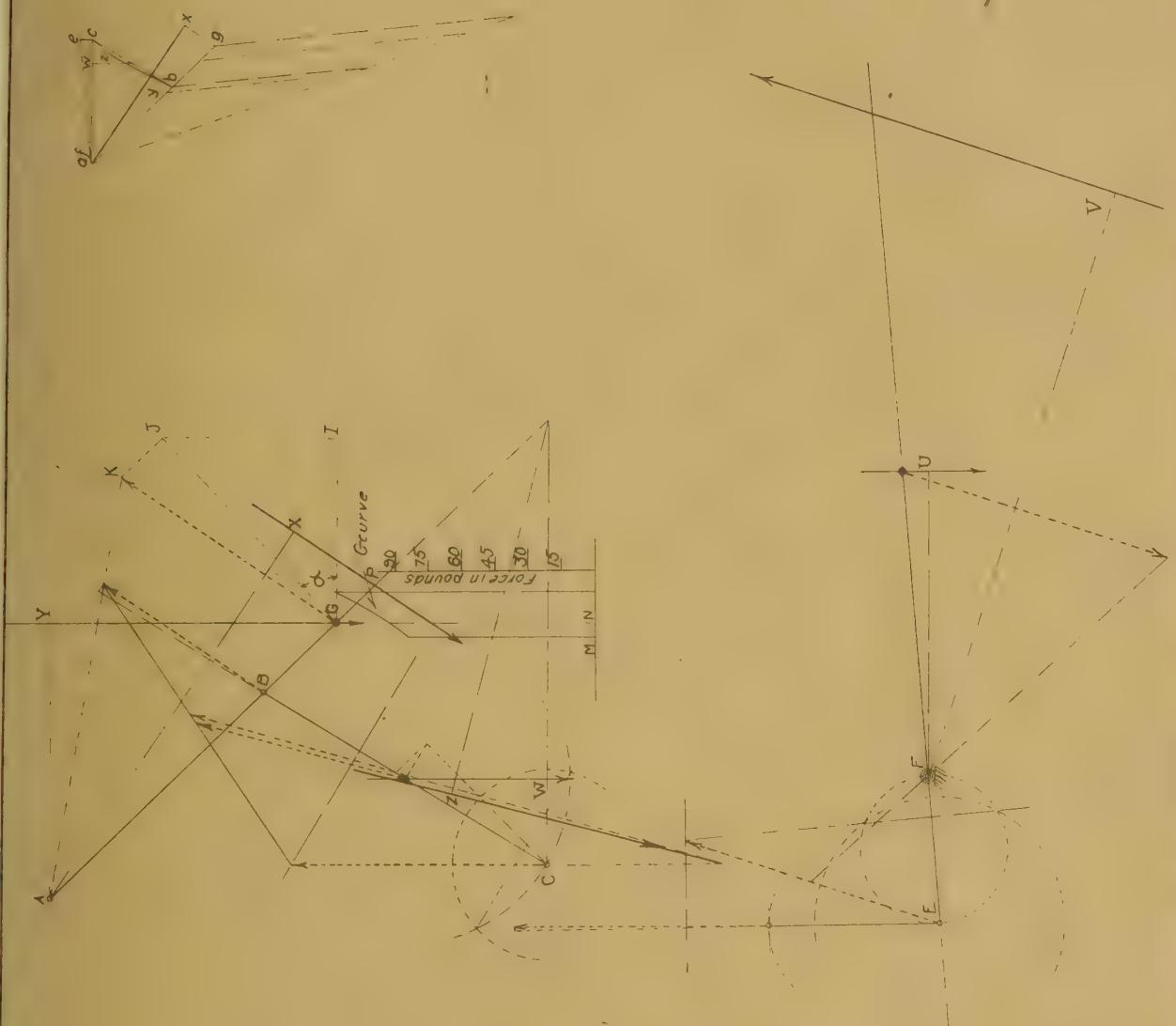
for the mechanism. Then lay off
GI equal to $\frac{d^2Z}{dt^2} = -.146$ feet per sec-
ond per second. From I erect a
perpendicular, IJ, then GJ is G's
tangential acceleration, G's radial
acceleration $= \frac{\overline{205}^2}{OG} = \frac{\overline{205}^2}{1} = .042$ feet
per second per second. Lay off
JK parallel to AG and equal to .042
feet per second per second, then GK
is the total acceleration of G. The
construction for the acceleration
of the remaining links of the
mechanism can now be made.
On Plate II, the heavy dotted lines
denote accelerations, the heavy solid
inertia forces and the light lines
the forces due to weight.

The acceleration of G = .19 feet
per second per second, mass of
link No. 1 is .37 gee pounds.

Therefore $F = .37 \times .19 = .0704$ pounds.

It was shown that this force passed

throughe a point a distance $\frac{k^2}{r}$ from the center of rotation and is parallel and opposite to G's acceleration

$$\frac{k^2}{r} = \frac{1.13}{1} = 1.13 \text{ feet}.$$

Lay of AP = 1.13 feet and draw XP parallel to G's acceleration, this is the action line of F.

Let F be the corresponding force reduced to G acting perpendicular to AG. The work of force F = F x velocity of application point X x dt. The work of reduced force F = F x velocity of g x dt.

Therefore

F x velocity of g · dt = F x velocity of X · dt

$$F = \frac{F \times \text{velocity of } X}{\text{velocity of } g}$$

Velocity of X = .228 feet per second.
Velocity of g = .205 feet per second.

$$F = \frac{.0704 \times .228}{.205} = .0785 \text{ pounds}.$$

The sign of this force is negative be-
cause it opposes the governor in its
motion. The line of action of the weight
is VG and w is the weight reduced
we have

$$w = \frac{W \times \text{velocity of } Y}{.205} = \frac{11.9 \times .142}{.205} = 8.28 \text{ lbs.}$$

The sign of this force is also negative
because it has a component opposite
to the direction of the motion of the
center of gravity. The remaining
forces were reduced in the same
way. The summary of the work is
given in table . The summation
of the reduced forces is − 70.0786
pounds. In obtaining this sum
the reduced of links 1 and 2 were
multiplied by 2 because there are
two of each of these links in the
governor mechanism. The action
line of these forces was taken
perpendicular to AG while the
problem called for reduced forces
with a horizontal action line. Let

TABLE IV

Link No.	Weight	Mass	Acceleration of Center of Gravity	Inertia Force	Velocity of Application Point	Reduced Force
1		.37	$.19 \frac{ft}{sec^2}$	$.0704^{\#}$	$.228 \frac{ft}{sec}$	$-.0785^{\#}$
1	11.90				.142	-8.28
2		.054	.159	.0086	.151	-.0063
2	1.74				.137	-.1163
3&4		2.94	.189	.555	.171	-.463
3&4	94.7				.171	-79.
5		.386	.413	.1595	.72	-.56
5	16.4				.36	+28.8

F_n be the required force then

$$F_n = \frac{F}{\cos \alpha} = \frac{-70.0786}{\cos \alpha} = -96 \text{ pounds}$$

The force scaled from the C curve for this position is 94 pounds which is sufficient to hold governor in equilibrium — leaving an additional force of 2 pounds to accelerate the governor masses.